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Research Report

ON THE REPRESENTATION OF DYNAMIC ALGEBRAS II

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ABSTRACT

A dynamic algebra is *representable* if it is represented by a standard Kripke model. In this paper we prove: (1) every separable dynamic algebra over an atomic Boolean algebra is representable, (2) there exists a countable dynamic algebra which is not representable, but (3) every countable separable dynamic algebra is the homomorphic image of a represenatable dynamic algebra, and moreover the homomorphism is obtained by reduction modulo inseparability. The last result is not true in general for uncountable algebras.

Introduction

In [K1] we defined *dynamic algebras* (see also [Pr1,Pr2]) and showed how they could be used to give a more algebraic interpretation to Propositional Dynamic Logic, in contrast to the more concrete Kripke model interpretation. The new semantics, although more amenable to techniques of universal algebra and model theory, involved a wider class of interpretations, and thus a potential loss of generality. In [K1] we asked how much loss of generality was involved. In that paper we showed that any dynamic algebra that satisfied a simple algebraic condition called *separability* is represented by a *possibly nonstandard* Kripke model (a *standard* model is one in which α^* is α^{rtc}, the reflexive transitive closure of α). The construction was similar to that of the Stone representation theorem for Boolean algebras (see [BS,Hal]). In fact, much of the useful algebraic/topological duality between Boolean algebras and Boolean spaces which grew out of Stone's work applies to dynamic algebras as well. In [K2] we developed these ideas by defining *dynamic spaces*, or Kripke models with a topology on the set of states satisfying certain properties, and established a duality between separable dynamic algebras and dynamic spaces. This duality is useful because it lets any problem be viewed from two different perspectives, and in some cases admits simpler proofs.

The representation theorem of [K1] left open the problem of whether every separable dynamic algebra is represented by a *standard* Kripke model. In [K3] we constructed a separable dynamic algebra not represented by any standard model, but the counterexample was uncountable and the technique did not extend to the countable case. It involved the construction of a dynamic space in which every point was a *nonstandard point*, i.e. was in some $<\alpha^*>X - <\alpha^{rtc}>X$ (standard Kripke models have no nonstandard points). However, in a dynamic space, any set $<\alpha^*>X - <\alpha^{rtc}>X$ is nowhere dense [K2]. Thus the counterexample must have been uncountable, since otherwise the set of nonstandard points would have been meager, and thus could not have comprised the entire space, by the Baire Category Theorem.

One of the results of this paper is the construction of a countable counterexample (this problem has been solved previously by Reiterman and Trnková [RT] in the absence of the $^-$ operator). However, in the countable case, we do have a positive representation result: any countable separable dynamic algebra is the homomorphic image of a countable representable dynamic algebra, and moreover the homomorphism is obtained by reduction modulo the congruence relation of inseparability. This is proved by using the Baire Category Theorem to show that, in a dynamic space with a countable algebra, a meager set of points containing all the nonstandard points can be deleted without changing the algebra. This result does not hold in general for uncountable algebras (it is false for the counterexample of [K3]) and explains the difficulty in finding a countable counterexample to the representation problem.

The proof uses a technical lemma with many other applications. It can be used to simplify the representation theorem of [K1]. We also use it here to prove that any separable dynamic algebra over an atomic Boolean algebra has a standard representation.

We assume familiarity with dynamic logic, dynamic algebras, and dynamic spaces. We use here the original definition of dynamic algebras [K1,K2] and not that of Pratt [Pr1,Pr2]. The following notation will be used: A *dynamic algebra* is a structure D = (**K**,**B**,<>) where **K** is a *Kleene algebra* (K,∪ , ; , ⁻ , * , 0 , λ), **B** is a *Boolean algebra* (B, ∨ , ∧ , ¬ , 0 , 1), and <> is a *scalar multiplication* **K** × **B** → **B** satisfying certain axioms (see [K1,K2]). Elements of **K** are denoted $\alpha,\beta,\ldots$ and elements of **B** are denoted X,Y,... ; the element <>(α,X) of **B** is denoted <α>X. A *nonstandard Kripke model* is a structure $A = (S,\mathbf{K},\mathbf{B})$, where **K** is a Kleene algebra of binary relations on S, **B** is a Boolean algebra of subsets of S, and all the Kleene algebra and Boolean algebra operations (with the possible exception of *) and the operation <> have their usual binary relation-theoretic and set-theoretic interpretations. The operator * applied to α gives the smallest element of **K** containing α^{rtc} (the reflexive transitive closure of α), but need not give α^{rtc} itself. A is *standard* if $\alpha^* = \alpha^{rtc}$. A *dynamic space* is a Kripke model (S,**K**,**B**) which is compact and Hausdorff when endowed with the topology on S generated by **B**, and in which all elements of **K** are closed in the product topology on S^2.

A technical lemma and applications

The following lemma appears implicitly in the proof of the representation theorem of [K1]. It gives a weak condition that is necessary and sufficient for a Kripke model-like structure to be a Kripke model and to have a given dynamic algebra as its characteristic algebra. After we prove the lemma, we will give two examples of its use. It can also be used to simplify the proof of the representation theorem of [K1].

Lemma 1. Let (**K**,**B**,<>) be a separable dynamic algebra and let ′ be a function from **B** onto a set **B**′ of subsets of a set S, and from **K** onto a set **K**′ of binary relations on S. If ′ is a Boolean algebra isomorphism **B** → **B**′ when **B**′ has the usual set-theoretic Boolean algebra operations, and if

$$(*) \qquad (<\alpha>X)' = <\alpha'>X' \text{ for all } \alpha \in \mathbf{K} \text{ and } X \in \mathbf{B}$$

where $<>$ has its usual Kripke model interpretation, then the structure (S,K',B') is a nonstandard Kripke model under the usual binary relation-theoretic Kleene algebra operations ; , $^-$, $\cup$, 0 , and λ, and $'$ is a dynamic algebra isomorphism $(K,B,<>) \rightarrow (K',B',<>)$.

Proof. First we observe that the structure $(K',B',<>)$ is separable: if $<\alpha'>X' = <\beta'>X'$ for all $X' \in B'$, then by (*), $(<\alpha>X)' = (<\beta>X)'$ for all $X \in B$. Since $'$ is one-one on B, $<\alpha>X = <\beta>X$ for all $X \in B$. Since $(K,B,<>)$ is separable, $\alpha = \beta$, and therefore $\alpha' = \beta'$.

A similar argument shows that $'$ is one-one on K: if $\alpha' = \beta'$, then $<\alpha'>X' = <\beta'>X'$ for all X', and by the argument of the above paragraph, $\alpha = \beta$.

Let us define the operator * on K' by $\alpha'^* = \alpha^{*\prime}$. We claim that under this definition, K' is a Kleene algebra and $'$ is a Kleene algebra homomorphism. First we show it is a morphism with respect to the ; operator. For all $X \in B$,

$$
\begin{aligned}
<(\alpha\beta)'>X' &= (<\alpha\beta>X)' \text{ by } (*), \\
&= (<\alpha>(<\beta>X))' \\
&= <\alpha'>(<\beta'>X') \text{ again by } (*), \\
&= <\alpha'\beta'>X' .
\end{aligned}
$$

Then $(\alpha\beta)' = \alpha'\beta'$ follows from separability. The proof for the operator $\cup$ is similar.

Since $'$ is one-one on K and is a morphism for $\cup$, $'$ preserves the semilattice order $\leq$ of K and K'. This says that * behaves in K' as desired, i.e. $\alpha'\beta'^*\gamma' = \sup_n \alpha'\beta'^n\gamma'$ and $<\alpha'^*>X' = \sup_n <\alpha'^n>X'$ for all X.

To show $\alpha^{-\prime} = \alpha'^-$, by separability it suffices to show that for all X, $<\alpha^{-\prime}>X' = <\alpha'^->X'$. The following proof of this fact depends strongly on the dynamic algebra axiom $X \leq [\alpha]<\alpha^->X$ and its dual $<\alpha>[\alpha^-]X \leq X$. Note that this holds in the structure $(K',B',<>)$, since $<>$ has its standard interpretation.

$$
\begin{aligned}
<\alpha^{-\prime}>X' &\leq <\alpha^{-\prime}>([\alpha']<\alpha'^->X') \\
&\leq <\alpha^{-\prime}>[\alpha'](<\alpha'^->X') \\
&\leq <\alpha'^->X' \\
&\leq <\alpha'^->([\alpha]<\alpha^->X)' \\
&\leq <\alpha'^->[\alpha'](<\alpha^{-\prime}>X') \\
&\leq <\alpha^{-\prime}>X' .
\end{aligned}
$$

Finally, (*) says that $'$ is a morphism for the operator $<>$. This completes the proof. □

Now we give two examples of the use of the above lemma. Corollary 2 gives a positive representation result for separable dynamic algebras over an atomic Boolean algebra. This is not true under Pratt's definition of dynamic algebras. Indeed, his example of Section 7 of [Pr2] is atomic.

Corollary 2. Any separable dynamic algebra over an atomic Boolean algebra has a standard representation.

Proof. Let $(\mathbf{K},\mathbf{B},<>)$ be such an algebra and let A be its dynamic space. A point s is *isolated* if $\{s\}$ is open; to say that $\mathbf{B}$ is atomic is the same as saying that the isolated points of A are dense [Hal]. Let S' be the set of isolated points. Let $A' = (S',\mathbf{K}',\mathbf{B}')$ be the structure obtained by restricting A to S'; i.e. $X' = X \cap S'$, $\alpha' = \alpha \cap S'^2$, $\mathbf{B}' = \{ X' \mid X \in \mathbf{B} \}$, and $\mathbf{K}' = \{ \alpha' \mid \alpha \in \mathbf{K} \}$.

We claim first that $(S',\mathbf{K}',\mathbf{B}')$ is a Kripke model and $(\mathbf{K}',\mathbf{B}',<>)$ is a dynamic algebra isomorphic to $(\mathbf{K},\mathbf{B},<>)$. Since all $X \in \mathbf{B}$ are open and S' is dense, if $X \neq 0$ then $X' \neq 0$. Therefore $\mathbf{B}$ and $\mathbf{B}'$ are isomorphic, thus we need only show the (*) condition of Lemma 1. Certainly $\supseteq$ holds. Now if $s \in (<\alpha>X)'$, then $s \in S'$ and $X \cap <\alpha^->\{s\} \neq 0$. Since $<\alpha^->$ preserves open sets [K2], $X \cap <\alpha^->\{s\}$ is open, hence must contain an isolated point t. Then $t \in X'$ and $(s,t) \in \alpha'$, i.e. $s \in <\alpha'>X'$.

It remains to prove that A' is standard. In any dynamic algebra, $X \wedge <\alpha^*>Y = \sup_n X \wedge <\alpha^n>Y$. Since $\{s\} \in \mathbf{B}$ for any isolated point s, if s, t are isolated then $\{s\} \cap <\alpha^*>\{t\} \neq 0$ implies $\{s\} \cap <\alpha^n>\{t\} \neq 0$ for some n, or in other words, $(s,t) \in \alpha^*$ implies $(s,t) \in \alpha^n$ for some n. Thus $\alpha^* = \alpha^{rtc}$. □

The next application allows us to collapse points of a Kripke model with the same theory. Let $(S,\mathbf{K},\mathbf{B})$ be a Kripke model. For $s \in S$, let the *theory of* s be

$$\mathrm{Th}(s) = \{ X \mid s \in X \},$$

and let

$$S' = \{ \mathrm{Th}(s) \mid s \in S \}$$
$$X' = \{ \mathrm{Th}(s) \mid s \in X \}, \quad \mathbf{B}' = \{ X' \mid X \in \mathbf{B} \},$$
$$\alpha' = \{ (\mathrm{Th}(s),\mathrm{Th}(t)) \mid (s,t) \in \alpha \}, \quad \mathbf{K}' = \{ \alpha' \mid \alpha \in \mathbf{K} \}.$$

Corollary 3. $(\mathbf{K},\mathbf{B},<>)$ and $(\mathbf{K}',\mathbf{B}',<>)$ are isomorphic. Moreover, if $(S,\mathbf{K},\mathbf{B})$ is standard then $(S',\mathbf{K}',\mathbf{B}')$ is.

Remark. Given any Kripke model with separable dynamic algebra, this construction gives a Hausdorff representation of the algebra. The construction is similar to Mirkowska's construction of a *normalized model* [M] or Parikh's construction of a *canonical model* [Pa]. Although a Hausdorff representation can be obtained from the representation theorem of [K1], this result shows that standardness need not be sacrificed.

Proof. The conditions of Lemma 1 are easily verified, so (K,B,<>) and (K′,B′,<>) are isomorphic. If $(Th_1,Th_2) \in \alpha'^{*} = \alpha^{*\prime}$, then $\exists\,(s,t) \in \alpha^{*}$ such that $Th(s) \in Th_1$ and $Th(t) \in Th_2$. Since (K,B,<>) is standard, $(s,t) \in \alpha^n$ for some n. Thus $(Th_1,Th_2) \in \alpha^{n\prime}$. □

Let (S,K,B) and (S',K',B') be as in Corollary 3, and let (S'',K'',B'') be the dynamic space of (K,B,<>). Recall from [K1] that S'' is the set of ultrafilters of B. For each $s \in S$, Th(s) is an ultrafilter, and it is easily checked that the map $s \to Th(s)$ is an embedding $(S,K,B) \to (S'',K'',B'')$. Corollary 3 says that this embedding factors through (K′,B′,<>), thus (S'',K'',B'') is the compactification of (K′,B′,<>).

The representation of countable separable dynamic algebras

Let (K,B,<>) be a countable separable dynamic algebra and let $A = (S,K,B)$ be its dynamic space. Let α^{rtc} denote the reflexive transitive closure of α. A *nonstandard point* is an element of a set $<\alpha^{*}>X - <\alpha^{rtc}>X$ for some $\alpha \in K$, $X \in B$. As shown in [K2], $<\alpha^{*}>X$ is the topological closure of $<\alpha^{rtc}>X$. This is because one of the axioms of dynamic algebra states that $<\alpha^{*}>X$ is the supremum of the $<\alpha^n>X$, $n \geq 0$, with respect to the lattice order $\leq$ in B. Therefore each set $<\alpha^{*}>X - <\alpha^{rtc}>X$ is nowhere dense, so in a countable algebra the set of all nonstandard points is meager.

More generally, let $^{\circ}$ be an uniterpreted unary operator symbol, and let $C(^{\circ})$ denote a term over K and B with one free occurrence of $^{\circ}$. Construct a set of terms F inductively: $<\alpha^{\circ}>X \in F$ for any $\alpha \in K$, $X \in B$, and if $C(^{\circ}) \in F$ then $<\beta>(Y \wedge C(^{\circ})) \in F$ for any $\beta \in K$, $Y \in B$. If $C(^{\circ}) \in F$ then let $C(^{*})$ denote the subset of A obtained by replacing the $^{\circ}$ in $C(^{\circ})$ by *, and similarly for $C(^{rtc})$, $C(^{n})$. Then for every $C(^{\circ}) \in F$, $C(^{*}) - C(^{rtc})$ is nowhere dense. This is proved by showing that $C(^{*})$ is the supremum of the $C(^{n})$ and the topological closure of $C(^{rtc})$.

F is clearly countable, and so let $C_i(^{\circ})$ be the i^{th} element of F and let $N_i = C_i(^{*}) - C_i(^{rtc})$. Let $M = \cup_{i \geq 1} N_i$. Then M is a meager set containing all the nonstandard points.

We wish to use Lemma 1 to show that the points of M can be removed without changing the algebra. Let $S' = S - M$, and let let A' be the structure obtained by restricting A to S', as in Corollary 2.

Theorem 4. (K,B,<>) and (K′,B′,<>) are isomorphic.

Proof. Since M is meager, by the Baire Category Theorem, S' is dense in S. Thus **B** and **B′** are isomorphic, as in Corollary 2.

It remains to show the (*) condition of Lemma 1. The inclusion $\supseteq$ is trivial. Now suppose $s \in (\langle\alpha\rangle X)'$. Let $N_0 = 0$. Construct an infinite sequence $X_0 \supseteq X_1 \supseteq \ldots$ of elements of **B** inductively, such that for all i,

(1) $s \in \langle\alpha\rangle X_i$

(2) $s \notin \langle\alpha\rangle(X_i \wedge N_i)$.

For the basis, take $X_0 = X$. Now suppose (1) and (2) hold for X_i. If $s \notin \langle\alpha\rangle(X_i \wedge N_{i+1})$, then take $X_{i+1} = X_i$ and we are done. Otherwise, $s \in \langle\alpha\rangle(X_i \wedge N_{i+1})$ and thus $s \in \langle\alpha\rangle(X_i \wedge C_{i+1}(^*))$. Since $\langle\alpha\rangle(X_i \wedge C_{i+1}(^*))$ is a term in F and since $s \notin M$, $s \in \langle\alpha\rangle(X_i \wedge C_{i+1}(^{rtc}))$ and so $s \in \langle\alpha\rangle(X_i \wedge C_{i+1}(^n))$ for some n. Take $X_{i+1} = X_i \wedge C_{i+1}(^n)$. It is easily verified that in either case (1) and (2) are preserved.

Now for all i the set $\langle\alpha^-\rangle\{s\} \cap X_i$ is nonempty by (1). It is also closed, since {s} is, and since $\langle\alpha^-\rangle$ preserves closed sets [K2]. Since A is compact, the intersection of all these sets contains a point t, and $t \notin M$ since for all i, $\langle\alpha^-\rangle\{s\} \cap X_i$ is disjoint from N_i by (2). Since $t \in \langle\alpha^-\rangle\{s\} \cap X'$, $s \in \langle\alpha'\rangle X'$, as desired. □

Theorem 5. Every countable separable dynamic algebra is a homomorphic image of a countable representable dynamic algebra. Moreover, the homomorphism is obtained by reduction modulo the congruence relation of inseparability.

Proof. Let (K,B,<>) be a countable separable dynamic algebra. Let $A = (S,\mathbf{K},\mathbf{B})$ be a Kripke model for (K,B,<>) with no nonstandard points; A exists by the previous theorem. Let $\mathbf{K}_{std}$ be the standard Kleene algebra on the states of A generated by the elements of **K**; that is, * in $\mathbf{K}_{std}$ is reflexive transitive closure, and the other operations have their usual interpretation. We will prove that ($\mathbf{K}_{std}$,B,<>) is the required homomorphic preimage of (K,B,<>).

Let L be the free (term) algebra freely generated by elements of **K** under operations $\cup$, $^-$, $;$, *. If $\alpha \in L$, Let α_K and $\alpha_{K_{std}}$ denote the interpretation of α in **K** and $\mathbf{K}_{std}$, respectively.

We claim first that for all $\alpha \in L$ and $X \in \mathbf{B}$, $<\alpha_K>X = <\alpha_{K_{std}}>X$. This is proved by induction on the length of the term α. It is surely true for the generators, since $\alpha_K = \alpha_{K_{std}}$. The induction cases $\alpha = \beta\gamma$ or $\alpha = \beta\cup\gamma$ are easy. For $\alpha = \beta^-$, note that α is equivalent to a shorter term, namely the term obtained by replacing each generator in β with its reverse. Finally, for $\alpha = \beta^*$, we have that

$$<\beta^*{}_{K_{std}}>X = \cup_{n\geq 0} <\beta_{K_{std}}{}^n>X$$

since * is reflexive transitive closure in $\mathbf{K}_{std}$, and

$$<\beta^*{}_K>X = \cup_{n\geq 0} <\beta_K{}^n>X$$

since A contains no nonstandard points, and the right hand sides of the above two equations are equal by the induction hypothesis.

From the above we see that scalar multiplication by elements of $\mathbf{K}_{std}$ preserves **B**, therefore $(\mathbf{K}_{std},\mathbf{B},<>)$ is a dynamic algebra. Moreover, the function $f:\alpha_{K_{std}} \rightarrow \alpha_K$ is well-defined, for if $\alpha_{K_{std}} = \beta_{K_{std}}$ then for any $X \in \mathbf{B}$,

$$<\alpha_K>X = <\alpha_{K_{std}}>X = <\beta_{K_{std}}>X = <\beta_K>X,$$

and by separability, $\alpha_K = \beta_K$. At this point it is relatively easy to show that f is a dynamic algebra homomorphism $(\mathbf{K}_{std},\mathbf{B},<>) \rightarrow (\mathbf{K},\mathbf{B},<>)$, and that application of f to an element of $\mathbf{K}_{std}$ is exactly reduction modulo the congruence relation of inseparability with respect to **B**, i.e. for all $\alpha, \beta \in \mathbf{K}_{std}$,

$$f(\alpha) = f(\beta) \quad \text{iff} \quad \forall X <\alpha>X = <\beta>X .$$

This completes the proof. □

In contrast, this theorem does not hold for the uncountable algebra constructed in [K3], since all ultrafilters of this algebra are nonstandard (i.e. contain $<\alpha^*>X$ but no $<\alpha^n>X$ for some α and X) and it is easily shown that the homomorphic preimage of any nonstandard ultrafilter is nonstandard.

A counterexample

In this section we show that the result of the previous section cannot be improved: there exists a countable separable dynamic algebra which is not representable. Reiterman and Trnková [RT] have shown that a countable counterexample exists in the absense of the $^-$ operator. However, their counterexample is over an atomic Boolean algebra, and so cannot be extended to include $^-$, by Corollary 2.

Let Q be the set of rational numbers, and let $S = \{ e^{ir} \mid r \in \mathbf{Q} \}$, a countable dense subset of the complex unit circle. For each $r \in \mathbf{Q}$, let α_r be the binary relation on S that maps points counterclockwise around S through an angle r: $\alpha_r = \{ (u, ue^{ir}) \mid u \in S \}$. Let the Kleene algebra operations $\cup$, $^-$, ;, 0, λ have their usual interpretations. The following identities hold:

$$\alpha_r\alpha_s = \alpha_{r+s}$$
$$\alpha_r{}^n = \alpha_{nr}$$
$$\alpha_r{}^- = \alpha_{-r}$$
$$\lambda = \alpha_0 = \alpha_{2\pi k}\,, \quad k \in \mathbb{Z}\,.$$

For any set $A \subseteq S$, applying $\langle\alpha_r\rangle$ to A translates A through an angle of r. Thus $\langle\alpha_r\rangle A = \{ ue^{-ir} \mid u \in A \} \subseteq S$.

The * operator is defined as follows: $\lambda^* = 0^* = \lambda$ and $\alpha^* = S^2$ for all other α.

Let **K** be the Kleene algebra generated by the α_r under these operations. Then every element of **K** is either S^2 or of the form $\alpha_{r_1} \cup \ldots \cup \alpha_{r_k}$. Let **B** be the Boolean algebra generated by the half-open intervals

$$S \cap \{ e^{ir} \mid r \in \mathbb{R} \text{ and } s \le r < t \}\,,$$

$s,t \in \mathbf{Q}$.

All dynamic algebra axioms not mentioning * hold in $(\mathbf{K},\mathbf{B},\langle\rangle)$, since all operators other than * have their standard interpretations. To show that $(\mathbf{K},\mathbf{B},\langle\rangle)$ is a dynamic algebra, it remains to show that for $\alpha,\beta,\gamma \in \mathbf{K}$ and $X \in \mathbf{B}$,

$$\alpha\beta^*\gamma = \sup_n \alpha\beta^n\gamma\,,$$
$$\langle\beta^*\rangle X = \sup_n \langle\beta^n\rangle X\,.$$

This is clearly true for the trivial cases ($X = 0$, $\beta \in \{0,\lambda\}$, $\alpha = 0$, or $\gamma = 0$). In all other cases, since π and any nonzero rational r are linearly independent over **Q**,

$$\cup_{n\geq 0} <\alpha_r{}^n>\{u\} = \{ ue^{-inr} \mid n \in \mathbf{N} \}$$

is dense in the unit circle, therefore

$$\alpha\beta^*\gamma = \sup_n \alpha\beta^n\gamma = S^2 ,$$
$$<\beta^*>X = \sup_n <\beta^n>X = S .$$

Finally, $(\mathbf{K},\mathbf{B},<>)$ is separable since every element of **K** is either S^2 or of the form $\alpha_{r_1}\cup\ldots\cup\alpha_{r_k}$, and for any two distinct such representations, an $X \in \mathbf{B}$ can be taken small enough so as to separate the two.

We claim now that $(\mathbf{K},\mathbf{B},<>)$ is not represented by any standard model. With each ultrafilter U of **B** we associate a unique complex number f(U), namely the unique point contained in the intersection of the closures (in the plane) of the elements of U. f(U) lies on the unit circle but is not necessarily in S. Let $<\alpha_r>U$ denote the set $\{ <\alpha_r>X \mid X \in U \}$. Then $<\alpha_r>U$ is again an ultrafilter, and moreover, $f(<\alpha_r>U) = e^{-ir}f(U)$.

Suppose this algebra were represented by a standard Kripke model A. Let s be a point in A and let Th(s) denote the ultrafilter consisting of all elements of **B** containing s. Let r be any nonzero rational number, and let s,t be distinct points of A such that $(s,t) \in \alpha_r{}^*$. Since A is standard, $(s,t) \in \alpha_r{}^n$ for some n, or equivalently, $s \in <\alpha_{nr}>\{t\}$. Then s is contained in every element of the ultrafilter $<\alpha_{nr}>Th(t)$, so $Th(s) = <\alpha_{nr}>Th(t)$, and hence $f(Th(s)) = e^{-inr}f(Th(t))$.

Now $\alpha_r{}^* = \alpha_{-r}{}^*$ in **K**, so $(s,t) \in \alpha_{-r}{}^*$ as well, and hence $s \in <\alpha_{-mr}>\{t\}$ for some m. By an argument similar to the above, $f(Th(s)) = e^{imr}f(Th(t))$. Thus

$$f(Th(s))/f(Th(t)) = e^{-inr} = e^{imr}$$

which implies that $e^{i(m+n)r} = 1$, or $(m+n)r = 2\pi k$ for some k. But this is impossible since r and π are linearly independent. □

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