Some Notes on Rational Spaces

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Abstract. Set constraints are inclusions between expressions denoting sets of ground terms over a finite ranked alphabet Σ. Rational spaces are topological spaces obtained as spaces of runs of topological Σ-hypergraphs. They were introduced by Kozen in [Koz95], where the topological structure of the spaces of solutions to systems of set constraints was given in terms of rational spaces. In this paper we continue the investigation of rational spaces. We give a Myhill-Nerode like characterization of rational points, which in turn is used to re-derive results about the rational points of finitary rational spaces. We define congruences on Σ-hypergraphs, investigate their interplay with the Myhill-Nerode characterization, and finally we determine the computational complexity of some decision problems related to rational spaces.

1 Introduction

Set constraints are inclusions between expressions denoting sets of ground terms. They have been used extensively in program analysis and type inference for many years [AM91a, AM91b, Hei92, HJ90b, JM79, Mis84, MR85, Rey69, YO88]. Considerable recent effort has focussed on the computational complexity of the satisfiability problem [AKVW93, AKW95, AW92, BGW93, CP94a, CP94b, GTT93a, GTT93b, HJ90a, Ste94a]. Set constraints have also recently been used to define a constraint logic programming language over sets of ground terms that generalizes ordinary logic programming over an Herbrand domain [Koz94].

Set constraints exhibit a rich mathematical structure. There are strong connections to automata theory [GTT93a, GTT93b], type theory [KPS93, KPS94], first-order monadic logic [BGW93, CP94a], Boolean algebras with operators [JT51, JT52], and modal logic [Koz93]. There are algebraic and topological formulations, corresponding roughly to "soft" and "hard" typing respectively, which are related by Stone duality [Koz93, Koz95].

Many results in the literature on set constraints are topological in flavor. In [Koz95], Kozen defines rational spaces, develops the basic theory, and shows how many results in the literature can be re-derived by general topological principles.

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Kozen also gives a complete characterization of the sets of solutions to systems of set constraints in terms of rational spaces.

In this paper we continue the investigation of rational spaces. We give a Myhill-Nerode like characterization of rational points and, based on this characterization, we give a simple and direct proof that the rational points of a finitary rational space are dense [Koz95]. We show that the rational points in finitary rational spaces in some sense exactly capture the topological structure of the spaces. We investigate congruences in Σ-hypergraphs and their interplay with the Myhill-Nerode characterization. Congruences in rational spaces are strongly related to the notion of bisimulation [Mil89] in models of concurrency and a similar notion has appeared in [Koz94] in the context of efficient constraint solving. We also determine the computational complexity of some decision problems related to rational embeddings.

The rest of the paper is organized as follows. In §2 we review the basic definitions of Σ-hypergraphs and rational spaces. In §3, we give the Myhill-Nerode like characterization of rational points and related results. In §4, we define congruences on Σ-hypergraphs and in §5 we present the complexity results. Finally, in §6 we draw conclusions and discuss future work.

2 Preliminary Definitions

Let Σ be a finite ranked alphabet consisting of symbols f, each with an associated arity n. Symbols in Σ of arity 0, 1, 2, and n are called *nullary*, *unary*, *binary*, and *n-ary*, respectively. Nullary elements are denoted by $a, b, \ldots$ and are called *constants*. The set of elements of Σ of arity n is denoted Σ_n. In the sequel, the use of expressions of the form $f(t_1, \ldots, t_n)$ carries the implicit assumption that f is of arity n.

The set of *ground terms* over Σ is denoted T_Σ. It is the least set such that if $t_1, \ldots, t_n \in T_\Sigma$ and $f \in \Sigma_n$, then $f(t_1, \ldots, t_n) \in T_\Sigma$. If $X = \{x, y, \ldots\}$ is a set of variables, then $T_\Sigma(X)$ denotes the set of terms over Σ and X, considering elements in X as symbols of arity 0.

To avoid trivial cases, we assume that Σ always contains at least one constant and one symbol of arity greater than zero.

2.1 Hypergraphs

Let Σ be a fixed finite ranked alphabet.

Definition 1. A Σ-*hypergraph* is a pair $\mathcal{D} = (D, E)$, where D is a set of states and E is an indexed family of hyperedge relations

$$E_f : D^n \longrightarrow 2^D \ , \qquad n = \text{arity}(f) \ , \tag{1}$$

one for every $f \in \Sigma$. □

Hence, for constant a, E_a is a subset of D, and for unary g, E_g is a binary relation on D. When no confusion is possible, we may omit Σ—*e.g.*, we may refer to $\mathcal{D}$ as a hypergraph.

A hypergraph (D, E) is said to be *entire* if every $E_f(d_1, \ldots, d_n)$ is nonempty, *deterministic* if every $E_f(d_1, \ldots, d_n)$ is a singleton, and *unrestricted* if every $E_f(d_1, \ldots, d_n)$ is D.

Definition 2. A *run* of a hypergraph $\mathcal{D} = (D, E)$ is a map $\theta : T_\Sigma \longrightarrow D$ such that for all $f(t_1, \ldots, t_n) \in T_\Sigma$,

$$\theta(f(t_1, \ldots, t_n)) \in E_f(\theta(t_1), \ldots, \theta(t_n)) \ . \tag{2}$$

The set of runs of $\mathcal{D}$ is denoted $\mathcal{R}(\mathcal{D})$. □

2.2 Rational Spaces

We recall the basic definitions from [Koz95].

Definition 3. A *topological Σ-hypergraph* is a Σ-hypergraph $\mathcal{D} = (D, E)$, finite or infinite, endowed with a topology on D whose hyperedges

$$\{(d, d_1, \ldots, d_n) \mid d \in E_f(d_1, \ldots, d_n)\} \tag{3}$$

are closed in the product topology on D^{n+1}. □

Definition 4. A *space of runs* over Σ is the space $\mathcal{R}(\mathcal{D})$ of runs of a topological Σ-hypergraph $\mathcal{D}$, where the topology on $\mathcal{R}(\mathcal{D})$ is inherited from the product topology on D^{T_Σ}. The space $\mathcal{R}(\mathcal{D})$ is called *finitary* if D is finite. □

The product topology on D^{T_Σ} is the smallest topology such that all projections $\pi_t : D^{T_\Sigma} \longrightarrow D$, mapping θ to $\theta(t)$, are continuous. Hence, it is generated by the subbasic open sets

$$\{\theta \mid \theta(t) \in x\} \ , \quad t \in T_\Sigma \ , \quad x \text{ open in } D \ . \tag{4}$$

Recall that open sets in D^{T_Σ} are then obtained as arbitrary unions of finite intersections of subbasic open sets. The space $\mathcal{R}(\mathcal{D})$ of runs of $\mathcal{D}$ is a subspace of this space. The topology is thus generated by subbasic open sets (4) restricted to $\mathcal{R}(\mathcal{D})$.

Proposition 5. [Koz95] *If $\mathcal{D}$ is finite and discrete, then $\mathcal{R}(\mathcal{D})$ is a complete metric space (all Cauchy sequences converge) under the metric*

$$d(\eta, \eta') \ = \ 2^{-depth(t)} \ , \tag{5}$$

where t is a term of minimal depth on which η and η' differ, or 0 if no such term exists.

Definition 6. A *rational space* is a space of runs $\mathcal{R}(\mathcal{D})$ such that $\mathcal{D}$ is Hausdorff and compact. □

In [Koz95] it is proved that if $\mathcal{D}$ is Hausdorff and/or compact, then so is the space of runs $\mathcal{R}(\mathcal{D})$. Hence, every rational space is Hausdorff and compact. Also, if a rational space $\mathcal{R}(\mathcal{D})$ is finitary, then the topology on D is discrete.

A category of rational spaces is obtained by defining morphisms as rational maps.

Definition 7. Let $\mathcal{R}(\mathcal{D})$ and $\mathcal{R}(\mathcal{D}')$ be rational spaces over Σ. A *rational map* from $\mathcal{R}(\mathcal{D})$ to $\mathcal{R}(\mathcal{D}')$ is a function $\widehat{h} : \theta \mapsto h \circ \theta$ defined by a continuous map $h : \mathcal{D} \longrightarrow \mathcal{D}'$ such that

$$h(E_f^{\mathcal{D}}(d_1, \ldots, d_n)) \;\subseteq\; E_f^{\mathcal{D}'}(h(d_1), \ldots, h(d_n)) \; . \tag{6}$$

A rational map $\widehat{h} : \mathcal{R}(\mathcal{D}) \longrightarrow \mathcal{R}(\mathcal{D}')$ is called a *rational embedding* if it is one-to-one, and a *refinement* if it is bijective. □

Notice that $\widehat{h}$ can be one-to-one or bijective even though h is not one-to-one.

If $\mathcal{D} = (D, E)$ and $\mathcal{D}' = (D, E')$ are two hypergraphs over the same set of states D, and if $E_f(d_1, \ldots, d_n) \subseteq E'_f(d_1, \ldots, d_n)$ for all $f \in \Sigma$ and $d_1, \ldots, d_n \in D$, then the identity map on D induces an embedding $\mathcal{R}(\mathcal{D}) \longrightarrow \mathcal{R}(\mathcal{D}')$, and $\mathcal{R}(\mathcal{D})$ is called a *narrowing* of $\mathcal{R}(\mathcal{D}')$.

If $\mathcal{D} = (D, E)$ is the induced subhypergraph of $\mathcal{D}' = (D', E')$ on some subset $D \subseteq D'$, *i.e.*, if $E_f(d_1, \ldots, d_n) = E'_f(d_1, \ldots, d_n) \cap D$ for all $f \in \Sigma$ and $d_1, \ldots, d_n \in D$, then the inclusion map $D \longrightarrow D'$ induces an embedding $\mathcal{R}(\mathcal{D}) \longrightarrow \mathcal{R}(\mathcal{D}')$, and $\mathcal{R}(\mathcal{D})$ is called an *induced subspace* of $\mathcal{R}(\mathcal{D}')$.

Definition 8. A *rational subspace* of a rational space is any embedded image of another rational space. In other words, a subspace $\mathcal{R}$ of a rational space $\mathcal{R}(\mathcal{D}')$ is a rational subspace if there exists a rational space $\mathcal{R}(\mathcal{D})$ and a rational embedding $\widehat{h} : \mathcal{R}(\mathcal{D}) \longrightarrow \mathcal{R}(\mathcal{D}')$ such that $\mathcal{R} = \widehat{h}(\mathcal{R}(\mathcal{D}))$.

A rational subspace is *entire* if it is the image of a rational space defined on an entire hypergraph. □

Definition 9. A *rational point* of a rational space is a singleton rational subspace $\mathcal{R}$ that is the embedded image of a finitary rational space $\mathcal{R}(\mathcal{D})$. □

Without loss of generality we may assume that $\mathcal{R}(\mathcal{D})$ is a singleton in Definition 9.

3 Myhill-Nerode

In this section we give an alternative characterization of rational points. Based on this characterization we give a simple and direct proof that the rational points are dense in any finitary rational space. We then continue by showing that the rational points in finitary rational spaces in some sense capture the topological structure of the spaces exactly; namely, if a rational map $\widehat{h} : \mathcal{R}(\mathcal{D}) \longrightarrow \mathcal{R}(\mathcal{D}')$ between finitary rational spaces induces a bijection between their rational points, then the spaces are homeomorphic.

The Myhill-Nerode characterization is based on special ground terms, Σ-contexts.

Definition 10. Let $*$ be a symbol not in Σ. A *Σ-context* is a term in $T_\Sigma(\{*\})$ containing exactly one occurrence of $*$. We denote a context by $C[\,]$. Given a ground term $t \in T_\Sigma$ and a context $C[\,]$ we let $C[t]$ denote the ground term in T_Σ obtained by replacing $*$ in $C[\,]$ by t. □

Definition 11. Let $\mathcal{D} = (D, E)$ be a hypergraph, θ a run in $\mathcal{R}(\mathcal{D})$, and $\approx \subseteq D \times D$ a binary relation. The relation $\approx_\theta \subseteq T_\Sigma \times T_\Sigma$ is then defined by

$$t \approx_\theta t' \quad \text{iff} \quad \forall C[\,].\, \theta(C[t]) \approx \theta(C[t']) \tag{7}$$

□

It follows that if $\approx$ is an equivalence relation, then so is $\approx_\theta$. In fact, $\approx_\theta$ will be a congruence with respect to Σ, *i.e.*, if $t_1 \approx_\theta t'_1, \ldots, t_n \approx_\theta t'_n$, then $f(t_1, \ldots, t_n) \approx_\theta f(t'_1, \ldots, t'_n)$.

The following theorem is the Myhill-Nerode like characterization of rational points.

Theorem 12. *A run $\theta \in \mathcal{R}(\mathcal{D})$ is a rational point if and only if $=_\theta$ has finite index, where $=$ is the identity relation on $\mathcal{D}$'s states.*

Proof. Assume θ is a rational point. Let $\widehat{h} : \mathcal{R}(\mathcal{D}') \longrightarrow \mathcal{R}(\mathcal{D})$ be a witnessing rational embedding, defined by $h : D' \longrightarrow D$. Let $\mathcal{R}(\mathcal{D}') = \{\eta\}$ and let $\mathcal{D}''$ be the entire subhypergraph of $\mathcal{D}'$ induced by $\eta(T_\Sigma)$. Then, with a slight abuse of notation, we have $\eta \in \mathcal{R}(\mathcal{D}'')$. Since $\mathcal{R}(\mathcal{D}')$ is a singleton $\mathcal{D}''$ must be deterministic and $\mathcal{R}(\mathcal{D}'')$ a singleton. Notice that h induces a rational embedding from $\mathcal{R}(\mathcal{D}'')$ to $\mathcal{R}(\mathcal{D})$, witnessing that $\theta = \widehat{h}(\eta)$ is a rational point. Hence, we may assume without loss of generality that $\mathcal{D}'$ is deterministic. It follows that $t =_\eta t'$ if and only if $\eta(t) = \eta(t')$, and since $\mathcal{D}'$ is finite, $=_\eta$ must have finite index. Next, we conclude that $=_\eta$ refines $=_\theta$, because for any context $C[\,]$ and terms $t =_\eta t'$

$$\begin{aligned} \theta(C[t]) &= (\widehat{h}(\eta))(C[t]) \\ &= (\widehat{h}(\eta))(C[t']) \\ &= \theta(C[t']) \ . \end{aligned}$$

But then $=_\theta$ has finite index.

Conversely, assume that $=_\theta$ has finite index. Let $\mathcal{D}'$ be the hypergraph whose states D' are the equivalence classes of $=_\theta$ and whose hyperedges are given by

$$E_f(d_1, \ldots, d_n) = \{d \mid \exists\, t_1 \in d_1, \ldots, t_n \in d_n.\, d = [f(t_1, \ldots, t_n)]_{=_\theta}\} \ ,$$

where $[t]_{=_\theta}$ denotes the equivalence class of t in $=_\theta$. We claim that $\mathcal{D}'$ is deterministic.

- each $E_f(d_1, \ldots, d_n)$ is nonempty: pick $t_1, \ldots, t_n$ in $d_1, \ldots, d_n$, respectively, then $[f(t_1, \ldots, t_n)]_{=_\theta} \in E_f(d_1, \ldots, d_n)$.
- each $E_f(d_1, \ldots, d_n)$ contains at most one element: if $d, d' \in E_f(d_1, \ldots, d_n)$ and $d \neq d'$, then there exist $t_1, t_1' \in d_1, \ldots, t_n, t_n' \in d_n$ such that $d = [f(t_1, \ldots, t_n)]_{=_\theta}$ and $d' = [f(t_1', \ldots, t_n')]_{=_\theta}$—but $=_\theta$'s congruence properties implies $f(t_1, \ldots, t_n) =_\theta f(t_1', \ldots, t_n')$, contradicting $d \neq d'$.

It follows that $\mathcal{R}(\mathcal{D}')$ must be a singleton $\{\eta\}$. An inductive argument shows that $\eta(t) = [t]_{=_\theta}$. Let D' be endowed with the discrete topology. Since D' is finite, D' is Hausdorff and compact. Define $h : D' \longrightarrow D$ by $[t]_{=_\theta} \mapsto \theta(t)$. The mapping is well-defined and trivially continuous. For $d \in E_f(d_1, \ldots, d_n)$ assume $d = [f(t_1, \ldots, t_n)]_{=_\theta}$, where $t_1 \in d_1, \ldots, t_n \in d_n$. Then $h(d) = \theta(f(t_1, \ldots, t_n))$, $h(d_1) = \theta(t_1), \ldots, h(d_n) = \theta(t_n)$. Since $\theta(f(t_1, \ldots, t_n)) \in E_f(\theta(t_1), \ldots, \theta(t_n))$ by definition of θ we conclude $h(E_f(d_1, \ldots, d_n)) \subseteq E_f(h(d_1), \ldots, h(d_n))$, *i.e.*, $\widehat{h} : \mathcal{R}(\mathcal{D}') \longrightarrow \mathcal{R}(\mathcal{D})$ is a rational embedding and $\widehat{h}(\eta)$ is a rational point of $\mathcal{R}(\mathcal{D})$. Since $\theta(t) = h([t]_{=_\theta}) = (\widehat{h}(\eta))(t)$, θ is a rational point of $\mathcal{R}(\mathcal{D})$. ■

We continue by giving simple proofs of two results from [Koz95].

Theorem 13. *The rational points of any finitary rational space $\mathcal{R}(\mathcal{D})$ are dense.*

Proof. Recall that $\mathcal{R}(\mathcal{D})$ is a complete metric space under the metric (5). Let θ be any point of $\mathcal{R}(\mathcal{D})$, $\mathcal{D} = (D, E)$. We wish to show that there exist rational points arbitrarily close to θ.

Let $\mathcal{D}' = (D', E')$ be a deterministic narrowing of the induced subspace on $D' = \theta(T_\Sigma)$. Then $\mathcal{D}'$ is entire and deterministic. For $f \in \Sigma_n$ and $d_1, \ldots, d_n \in D'$, let $H_f(d_1, \ldots, d_n)$ be the unique element of $E_f'(d_1, \ldots, d_n)$.

For each $k \geq 0$, define inductively

$$\eta(f(t_1, \ldots, t_n)) = \begin{cases} \theta(f(t_1, \ldots, t_n)) \ , & \text{if } \mathit{depth}(f(t_1, \ldots, t_n)) \leq k \\ H_f(\eta(t_1), \ldots, \eta(t_n)) \ , & \text{otherwise.} \end{cases}$$

Then $\eta \in \mathcal{R}(\mathcal{D})$, since if $\mathit{depth}(f(t_1, \ldots, t_n)) \leq k$, then

$$\begin{aligned} \eta(f(t_1, \ldots, t_n)) &= \theta(f(t_1, \ldots, t_n)) \\ &\in E_f(\theta(t_1), \ldots, \theta(t_n)) \\ &= E_f(\eta(t_1), \ldots, \eta(t_n)) \ , \end{aligned}$$

and if $\mathit{depth}(f(t_1, \ldots, t_n)) > k$, then

$$\begin{aligned} \eta(f(t_1, \ldots, t_n)) &= H_f(\eta(t_1), \ldots, \eta(t_n)) \\ &\in E_f'(\eta(t_1), \ldots, \eta(t_n)) \\ &\subseteq E_f(\eta(t_1), \ldots, \eta(t_n)) \ . \end{aligned}$$

The point η is of distance at most 2^{-k} from θ, since it agrees with θ on all terms of depth at most k.

Finally, we show that η is a rational point. If $depth(s)$, $depth(t) > k$ and $\eta(s) = \eta(t)$, then for all contexts $C[\,]$, $\eta(C[s]) = \eta(C[t])$. This can be shown by induction on the structure of $C[\,]$. Basis, $C[\,] = *$:

$$\eta(*[s]) \;=\; \eta(s) \;=\; \eta(t) \;=\; \eta(*[t])\ ,$$

Induction step, $C[\,] = f(t_1, \ldots, t_{i-1}, C'[\,], t_{i+1}, \ldots, t_n)$:

$$\begin{aligned}\eta(C[s]) &= \eta(f(t_1, \ldots, t_{i-1}, C'[s], t_{i+1}, \ldots, t_n))\\ &= H_f(\eta(t_1), \ldots, \eta(t_{i-1}), \eta(C'[s]), \eta(t_{i+1}), \ldots, \eta(t_n))\\ &= H_f(\eta(t_1), \ldots, \eta(t_{i-1}), \eta(C'[t]), \eta(t_{i+1}), \ldots, \eta(t_n))\\ &= \eta(f(t_1, \ldots, t_{i-1}, C'[t], t_{i+1}, \ldots, t_n))\\ &= \eta(C[t])\end{aligned}$$

It follows from Theorem 12 that η is a regular point, since there are only finitely many terms of depth k or less, and these account for finitely many $=_\eta$-classes; and for terms t of depth greater than k, the above argument shows that the $=_\eta$-class is determined by $\eta(t)$. Thus $=_\eta$ is of finite index. ■

Corollary 14. *Every nonempty finitary rational space contains a rational point.*

Next, we continue by showing how rational points of finitary rational spaces capture the topology of the spaces.

3.1 Rational Points and The Topology

In this section we show how the rational points of finitary rational spaces capture the topological structure of the space.

Rational maps always preserve rational points. In fact, for finitary rational spaces injectivity can be determined by looking at rational points only.

Lemma 15. *Let $\widehat{h} : \mathcal{R}(\mathcal{D}) \longrightarrow \mathcal{R}(\mathcal{D}')$ be a rational map between finitary rational spaces. If $\gamma, \theta \in \mathcal{R}(\mathcal{D})$ are distinct runs such that $\widehat{h}(\gamma) = \widehat{h}(\theta)$, then there exist distinct rational points $\eta_1, \eta_2 \in \mathcal{R}(\mathcal{D})$ such that $\widehat{h}(\eta_1) = \widehat{h}(\eta_2)$.*

Proof. Assume γ and θ are the above mentioned runs. Let

$$P \;=\; \{\, (d_1, d_2) \mid \text{there exist infinitely many } t \text{ such that } (\gamma(t), \theta(t)) = (d_1, d_2) \,\}\ .$$

Since $\mathcal{D}$ is finite the set P is finite and hence there must exist natural numbers $0 < n$ and $0 < k$ such that

- $\exists t_0.\, depth(t_0) < n \wedge \gamma(t_0) \neq \theta(t_0)$
- $\forall t.\, depth(t) \geq n \Rightarrow (\gamma(t), \theta(t)) \in P$
- $\forall (d_1, d_2) \in P.\, \exists t.\, n \leq depth(t) < n + k \wedge (\gamma(t), \theta(t)) = (d_1, d_2)$

For each $(d_1, d_2) \in P$ there exists by definition a term $r_{(d_1,d_2)}$ such that $n \leq depth(r_{(d_1,d_2)}) < n + k$ and $(\gamma(r_{(d_1,d_2)}), \theta(r_{(d_1,d_2)})) = (d_1, d_2)$. Let $link : T_\Sigma \longrightarrow T_\Sigma$ be the partial function defined by

– if $depth(t) < n + k$, then $link(t) = t$
– if $depth(t) = n + k$, then $link(t) = r_{(\gamma(t),\theta(t))}$
– if $depth(t) > n + k$, then $link(t)$ is undefined

Let η_1 and η_2 be partial functions from T_Σ to D defined by

– if $depth(t) \leq n + k$, then $\eta_1(t) = \gamma(t)$, else undefined
– if $depth(t) \leq n + k$, then $\eta_2(t) = \theta(t)$, else undefined

We will simultaneously extend the domain of definition of the functions $link, \eta_1$, and η_2 by defining them on terms of increasing depth, starting by depth $n+k+1$. We will maintain the invariant

1. $link, \eta_1$, and η_2 have same domain of definition, namely all terms up to a certain depth
2. if $t \in dom(\eta_1)$, then
   2.1. if $depth(t) \geq n$, then $(\eta_1(t), \eta_2(t)) \in P$
   2.2. $depth(t) \geq depth(link(t))$ and if $depth(t) \geq n + k$, then $link(t) = r_{(\eta_1(t),\eta_2(t))}$
   2.3. if $depth(f(t_1, \ldots, t_n)) > n + k$, then $\eta_i(f(t_1, \ldots, t_n)) = \eta_i(f(link(t_1), \ldots, link(t_n)))$, for $i = 1, 2$
   2.4. $(\eta_1(t), \eta_2(t)) = (\eta_1(link(t)), \eta_2(link(t)))$
   2.5. $h(\eta_1(t)) = h(\eta_2(t))$
3. if $f(t_1, \ldots, t_n) \in dom(\eta_1)$, then $\eta_i(f(t_1, \ldots, t_n)) \in E_f(\eta_i(t_1), \ldots, \eta_i(t_n))$, for $i = 1, 2$

Assume $link, \eta_1$, and η_2 have been defined on exactly all terms of depth less than $m > n + k$, and that the invariant holds. Pick a term $t = f(t_1, \ldots, t_n)$ of depth m. Let $t'_1 = link(t_1), \ldots, t'_n = link(t_n)$, and $t' = f(t'_1, \ldots, t'_n)$. Since $depth(t) > n+k$ we conclude $n \leq depth(t') \leq n+k$, by 2.2. Then, $(\eta_1(t'), \eta_2(t')) = (\gamma(t'), \theta(t')) \in P$, by the definitions of η_1, η_2, and n. Let $link(t) = r_{(\gamma(t'),\theta(t'))}$ and $(\eta_1(t), \eta_2(t)) = (\gamma(t'), \theta(t'))$.

Having defined $link, \eta_1$, and η_2 for all terms of depth m, claim that the invariant is maintained. We need only consider terms t of depth m. Clearly, 1. holds. From the above, 2.1., 2.2., and 2.3. follow easily. Next, observe that 2.4. follows from 2.1., 2.2., the fact that $(\eta_1(r), \eta_2(r)) = (\gamma(r), \theta(r))$ for terms r with $depth(r) < n + k$, and that $(\gamma(r_{(d_1,d_2)}), \theta(r_{(d_1,d_2)})) = (d_1, d_2)$. 2.5. follows from 2.1. and $\widehat{h}(\gamma) = \widehat{h}(\theta)$. To see that 3. holds, let $t = f(t_1, \ldots, t_n)$ and $t' = f(t'_1, \ldots, t'_n)$ as above. Then

$$\begin{aligned} \eta_1(t) = \eta_1(t') &\in E_f(\eta_1(t'_1), \ldots, \eta_1(t'_n)) \ , \quad \text{(3. used on } t'\text{)} \\ &= E_f(\eta_1(link(t_1)), \ldots, \eta_1(link(t_n))) \\ &= E_f(\eta_1(t_1), \ldots, \eta_1(t_n)) \ , \quad \text{(2.4. used on } t_1, \ldots, t_n\text{)} \end{aligned}$$

A similar argument holds for $\eta_2(t)$. Let η_1 and η_2 denote the total functions obtained by considering the limit to infinity of the above construction. By 3.,

$\eta_1, \eta_2 \in \mathcal{R}(\mathcal{D})$ and by 2.5., $\widehat{h}(\eta_1) = \widehat{h}(\eta_2)$. Also, $\eta_1 \neq \eta_2$, since they differ on the term t_0.

We conclude by showing that η_1 and η_2 are rational points. Let the *height* of a context $C[\ ] \in T_\Sigma(\{*\})$ be the depth of the element $*$. By induction in the height of the context $C[\ ]$, we show that if t and t' are terms of depth $n+k$ or greater, then $(\eta_1(t), \eta_2(t)) = (\eta_1(t'), \eta_2(t'))$ implies $(\eta_1(C[t]), \eta_2(C[t])) = (\eta_1(C[t']), \eta_2(C[t']))$.

For the context of height 0 there is nothing to prove, so assume $C[\ ]$ is a context of height $l > 0$. Then there exist contexts $C'[\ ]$ and $C''[\ ]$ of height 1 and $l-1$, respectively, such that $C[t] = C'[C''[t]]$ and $C[t'] = C'[C''[t']]$. By induction $(\eta_1(C''[t]), \eta_2(C''[t])) = (\eta_1(C''[t']), \eta_2(C''[t']))$ and by 2.2. $\mathit{link}(C''[t]) = \mathit{link}(C''[t'])$. Without loss of generality, let $C'[\ ] = g(t_1, \ldots, t_{i-1}, *, t_{i+1}, \ldots, t_m)$, where $g \in \Sigma_{m>0}$, $1 \leq i \leq m$, and $t_1, \ldots, t_{i-1}$, $t_{i+1}, \ldots, t_m$ are arbitrary terms. Then, by 2.3.

$$\begin{aligned}
\eta_1(C[t]) &= \eta_1(g(t_1, \ldots, t_{i-1}, C''[t], t_{i+1}, \ldots, t_m)) \\
&= \eta_1(g(\mathit{link}(t_1), \ldots, \mathit{link}(t_{i-1}), \mathit{link}(C''[t]), \mathit{link}(t_{i+1}), \ldots, \mathit{link}(t_m))) \\
&= \eta_1(g(\mathit{link}(t_1), \ldots, \mathit{link}(t_{i-1}), \mathit{link}(C''[t']), \mathit{link}(t_{i+1}), \ldots, \mathit{link}(t_m))) \\
&= \eta_1(g(t_1, \ldots, t_{i-1}, C''[t'], t_{i+1}, \ldots, t_m)) \\
&= \eta_1(C[t'])
\end{aligned}$$

Similarly, $\eta_2(C[t]) = \eta_2(C[t'])$. Hence, $t =_{\eta_1} t'$ and $t =_{\eta_2} t'$. Since there are only finitely many terms of depth less than $n+k$ and only finitely many elements in P, we conclude that $=_{\eta_1}$ and $=_{\eta_2}$ have finite index. By Theorem 12, η_1 and η_2 are rational points. ∎

The main result in this section is the following theorem.

Theorem 16. *Let $\widehat{h} : \mathcal{R}(\mathcal{D}) \longrightarrow \mathcal{R}(\mathcal{D}')$ be a rational map between finitary rational spaces. Assume $\widehat{h}$ is a bijection between the rational points of the spaces. Then $\widehat{h}$ is a homeomorphism.*

Proof. By Lemma 15 and the fact that $\widehat{h}(\theta)$ is a rational point if θ is rational point, we conclude that $\widehat{h}$ must be one-to-one.

Recall that any finitary rational space is a complete metric space under the metric

$$d(\eta, \eta') \;=\; 2^{-depth(t)} \;,$$

where t is a term of minimal depth on which η and η' differ, or 0 if no such term exists. Let $\theta \in \mathcal{R}(\mathcal{D}')$. By Theorem 13 there exist a sequence of rational points $\theta_1, \theta_2, \ldots \in \mathcal{R}(\mathcal{D}')$ converging to θ such that $d(\theta, \theta_i) < 2^{-i}$, for $i = 1, 2, \ldots$. Let $\{\eta_i\} = \widehat{h}^{-1}(\theta_i)$, for $i = 1, 2, \ldots$. Since D is finite, there must be infinitely many η_i's that agree on the finitely many terms of depth 1. Let $\eta_{i_{(1,1)}}, \eta_{i_{(1,2)}}, \ldots$ be such an infinite subsequence of $\eta_1, \eta_2, \ldots$. Let $\gamma_1 = \eta_{i_{(1,1)}}$. By a similar argument, there must be an infinite subsequence $\eta_{i_{(2,1)}}, \eta_{i_{(2,2)}}, \ldots$ of $\eta_{i_{(1,2)}}, \eta_{i_{(1,3)}}, \ldots$ that agree on all terms of depth at most 2. Let $\gamma_2 = \eta_{i_{(2,1)}}$. Repeating this procedure we obtain the infinite subsequence $\gamma_1, \gamma_2, \ldots$ of $\eta_1, \eta_2, \ldots$. This sequence is a Cauchy sequence. Let γ denote the run this sequence converges to. Notice $\gamma_1, \gamma_2, \ldots$ is mapped under $\widehat{h}$ to the infinite subsequence $\theta_{i_{(1,1)}}, \theta_{i_{(2,1)}}, \ldots$ of $\theta_1, \theta_2, \ldots$, which also converges to θ. Since $\widehat{h}$ is continuous γ must be mapped to θ. We conclude that $\widehat{h}$ is onto.

Since any rational space is compact and Hausdorff and $\widehat{h}$ is bijective we conclude that $\widehat{h}$ is a homeomorphism. ■

Remark. Notice that in general, if $f : S_1 \longrightarrow S_2$ is a continuous function between compact, complete metric topological spaces, such that f is a bijection between a dense subset of S_1 and a dense subset of S_2, then f may not even be a bijection, and hence, S_1 and S_2 not homeomorphic. For example, consider the interval $[0;1]$ and the circle C obtained by "gluing" the endpoints of $[0;1]$ together. Their topology is given by the usual Euclidean metric. The obvious mapping f from $[0;1]$ to C defined by mapping $0 \leq x < 1$ to x in C and 1 to 0 in C is continuous. Moreover, the rational numbers in $[0;1[$ are dense in $[0;1]$ and their image under f is dense in C. However, f is a bijection between these dense subsets, but not between $[0;1]$ and C.

Proposition 17. *Let $\widehat{h} : \mathcal{R}(\mathcal{D}) \longrightarrow \mathcal{R}(\mathcal{D}')$ be a rational map between finitary rational spaces. If $\widehat{h}$ is not onto, then there exists a rational point $\theta \in \mathcal{R}(\mathcal{D}')$ not in the image of $\widehat{h}$.*

Proof. The proof is similar to that of Theorem 16. ■

4 Congruences

In this section we define congruences on hypergraphs. We then investigate the relationship to special rational maps, so-called full homomorphisms, and to the Myhill-Nerode characterization from the previous section.

Definition 18. Let $\mathcal{D} = (D, E)$ be a hypergraph. A $\mathcal{D}$-*bisimulation* is a reflexive, symmetric relation $\approx \subseteq D \times D$ such that whenever $d \approx d'$, then

$$\forall\, a \in \Sigma_0.\, d \in E_a \;\Leftrightarrow\; d' \in E_a \tag{8}$$

$$\begin{aligned} &\forall\, f \in \Sigma_{n>0}.\, \forall 1 \leq i \leq n.\, \forall d_1, \ldots, d_{i-1}, d_{i+1}, \ldots, d_n. \\ &\quad \forall\, d'' \in E_f(d_1, \ldots, d_{i-1}, d, d_{i+1}, \ldots, d_n). \\ &\quad \exists\, d''' \in E_f(d_1, \ldots, d_{i-1}, d', d_{i+1}, \ldots, d_n).\, d'' \approx d''' \;. \end{aligned} \tag{9}$$

A $\mathcal{D}$-*congruence* is a $\mathcal{D}$-bisimulation that is an equivalence relation. □

The identity relation on D is a $\mathcal{D}$-congruence and the largest $\mathcal{D}$-congruence is given by

$$\bigcup\{ \approx \mid \approx \text{ is a } \mathcal{D}\text{-congruence } \} .$$

Notice that if Σ only contains constant and unary symbols, then the constants can be seen as state labels while the unary symbols can be seen as edge labels. In this case the largest $\mathcal{D}$-congruence, $\sim$, has the following relation to Milner's strong bisimulation [Mil89]: $\sim$ is the largest strong bisimulation with the additional property that $d \sim d'$ if and only if d and d' labeled identically (with respect to the constants).

Definition 19. Let $\mathcal{D} = (D, E)$ be a hypergraph and $\approx$ be a $\mathcal{D}$-congruence. $\mathcal{D}/\approx$ is the hypergraph $(D/\approx, E/\approx)$ given by

$$D/\approx = \{[d]_\approx \mid d \in D\} \tag{10}$$

$$E/\approx_f ([d_1]_\approx, \dots, [d_n]_\approx) = \{[d]_\approx \mid d \in E_f(d_1, \dots, d_n)\} . \tag{11}$$

□

Notice that the hyperedge relations are well-defined because $d_1 \approx d'_1, \dots, d_n \approx d'_n$ implies $\{[d]_\approx \mid d \in E_f(d_1, \dots, d_n)\} = \{[d]_\approx \mid d \in E_f(d'_1, \dots, d'_n)\}$.

Definition 20. Given Σ-hypergraphs $\mathcal{D}$ and $\mathcal{D}'$. A mapping $h : D \longrightarrow D'$ is a *homomorphism from* $\mathcal{D}$ if

$$\forall a \in \Sigma_0.\, h^{-1}(E'_a) = E_a \tag{12}$$

$$\forall f \in \Sigma_{n>0}.\, \forall d_1, \dots, d_n \in D.\, h(E_f(d_1, \dots, d_n)) = E'_f(h(d_1), \dots, h(d_n)) \tag{13}$$

The homomorphism h is *full*, if $h(D) = D'$. Two full homomorphisms $h_1 : D \longrightarrow D_1$ and $h_2 : D \longrightarrow D_2$ are *equivalent* if $\mathcal{D}_1$ and $\mathcal{D}_2$ are isomorphic under some $f : D_1 \longrightarrow D_2$ such that $h_2 = f \circ h_1$. □

Proposition 21. *Given $\mathcal{D}$. The $\mathcal{D}$-congruences are in one-to-one correspondence with the full homomorphisms, up to equivalence, from $\mathcal{D}$.*

Proof. Given a $\mathcal{D}$-congruence $\approx$. Define $[\,]_\approx : D \longrightarrow D/\approx$ as the mapping $d \mapsto [d]_\approx$. By Definition 18 and 19 it follows easily that $[\,]_\approx$ is a full homomorphism.

Conversely, given $\mathcal{D}'$ and a homomorphism $h : D \longrightarrow D'$. For $d, d' \in D$ define $d \approx_h d'$ if $h(d) = h(d')$. We claim $\approx_h$ is a $\mathcal{D}$-congruence. Clearly, $\approx_h$ is an equivalence relation. Also, if $d \in E_a$ for some $a \in \Sigma_0$, then $h(d) = h(d') \in E'_a$, by (12), hence $d' \in E_a$. If $f \in \Sigma_{n>0}$, $1 \le i \le n$, $d_1, \dots, d_{i-1}, d_{i+1}, \dots, d_n \in D$, and $d'' \in E_f(d_1, \dots, d_{i-1}, d, d_{i+1}, \dots, d_n)$, then

$$\begin{aligned} h(d'') &\in h(E_f(d_1,\ldots,d_{i-1},d,d_{i+1},\ldots,d_n)) \\ &= E'_f(h(d_1),\ldots,h(d_{i-1}),h(d),h(d_{i+1}),\ldots,h(d_n)) \\ &= E'_f(h(d_1),\ldots,h(d_{i-1}),h(d'),h(d_{i+1}),\ldots,h(d_n)) \\ &= h(E_f(d_1,\ldots,d_{i-1},d',d_{i+1},\ldots,d_n)) \ . \end{aligned}$$

So there must exist a $d''' \in E_f(d_1,\ldots,d_{i-1},d',d_{i+1},\ldots,d_n)$ such that $h(d'') = h(d''')$, *i.e.*, $d'' \approx_h d'''$. Hence, $\approx_h$ is a $\mathcal{D}$-congruence.

The mapping that maps a $\mathcal{D}$-congruence $\approx$ to the full homomorphism $[\,] : D \longrightarrow D/\approx$ is clearly one-to-one. Conversely, given any full homomorphism $h : D \longrightarrow D'$, $\mathcal{D}/\approx_h$ is isomorphic to $\mathcal{D}'$ under the mapping $[d]_{\approx_h} \mapsto h(d)$. ■

Given a run $\theta \in \mathcal{R}(\mathcal{D}/\approx)$, define η_θ inductively in t such that $\eta_\theta(t) \in \theta(t)$ as follows: assume $t = f(t_1,\ldots,t_n)$ and that $\eta_\theta(t_1),\ldots,\eta_\theta(t_n)$ have been defined. Since θ is a run, $\theta(t) \in E/\approx_f (\theta(t_1),\ldots,\theta(t_n))$. By the definition of $\mathcal{D}/\approx$, $\theta(t) = [d]_\approx$, where $d \in E_f(d_1,\ldots,d_n)$ and $d_1 \in \theta(t_1),\ldots,d_n \in \theta(t_n)$. Inductively we may assume that $\eta_\theta(t_1) \in \theta(t_1),\ldots,\eta_\theta(t_n) \in \theta(t_n)$, so $\eta_\theta(t_1) \approx d_1,\ldots,\eta_\theta(t_n) \approx d_n$. Since $\approx$ is a $\mathcal{D}$-congruence we conclude that there exists a $d' \in E_f(\eta_\theta(t_1),\ldots,\eta_\theta(t_n))$ such that $d \approx d'$. Define $\eta_\theta(t) = d'$. Then $\eta_\theta(t) \in \theta(t)$. It is easy to see that η_θ is indeed a run of $\mathcal{D}$. We refer to η_θ as a run *extracted* from θ. Notice that if η_θ is extracted from θ, then the mapping $t \mapsto [\eta_\theta(t)]_\approx$ equals θ. Also,

$$\begin{aligned} t \approx_{\eta_\theta} t' &\text{ iff } \forall C[\,].\, \eta_\theta(C[t]) \approx \eta_\theta(C[t']) \\ &\text{ iff } \forall C[\,].\, \theta(C[t]) = \theta(C[t'])\ , \quad (\eta_\theta(r) \in \theta(r) \text{ for any } r) \\ &\text{ iff } t =_\theta t'\ , \end{aligned}$$

i.e., $\approx_{\eta_\theta} \,=\, =_\theta$. Conversely, if $\eta \in \mathcal{R}(\mathcal{D})$, then the mapping $\theta_\eta : T_\Sigma \longrightarrow D/\approx$ given by $t \mapsto [\eta(t)]_\approx$ is a run of $\mathcal{R}(\mathcal{D}/\approx)$. Since $\approx$ is reflexive, we always have $=_\eta \subseteq \approx_\eta$ for any run in $\mathcal{R}(\mathcal{D})$. However, if $\sim$ is the largest $\mathcal{D}$-congruence and $\mathcal{D}'$ is $\mathcal{D}/\sim$, then it can be shown that the only $\mathcal{D}'$-congruence is the identity relation.

The construction of $\approx_\theta$ from Definition 11 induces an equivalence relation on $\mathcal{R}(\mathcal{D})$ as follows. It will be notationally convenient to "overload" the symbol $\approx$.

Definition 22. Let $\mathcal{D} = (D, E)$ be a hypergraph and $\approx \subseteq D \times D$ be a relation. For runs $\eta, \theta \in \mathcal{R}(\mathcal{D})$ define

$$\eta \approx \theta \ \text{ iff } \ \approx_\eta = \approx_\theta \ . \tag{14}$$

□

The following theorem exhibits a bijective correspondence between equivalence classes over runs of hypergraphs.

Theorem 23. *Let $\mathcal{D} = (D, E)$ be a hypergraph. Let $\sim$ be a $\mathcal{D}$-congruence. Then the mapping from $\mathcal{R}(\mathcal{D}/{\sim})/{=}$ to $\mathcal{R}(\mathcal{D})/{\sim}$ defined by $[\theta]_= \mapsto [\eta_\theta]_\sim$, where η_θ is a run extracted from θ, is well-defined. Moreover, it is a bijection.*

Proof. To see that the described mapping is well-defined assume that η, γ are two runs extracted from θ_1 and θ_2, respectively, where $\theta_1, \theta_2 \in [\theta]_= \in \mathcal{R}(\mathcal{D}/{\sim})/{=}$. We show that they are mapped to the same element of $\mathcal{R}(\mathcal{D})/{\sim}$, *i.e.*, that $[\eta]_\sim = [\gamma]_\sim$. By definition this means ${\sim_\eta} = {\sim_\gamma}$. We show $t \sim_\eta t'$ if and only if $t \sim_\gamma t'$. From the definitions we get

$$\begin{aligned} t \sim_\eta t' &\text{ iff } t =_{\theta_1} t' \\ &\text{ iff } t =_{\theta_2} t' \ , \quad (\theta_1, \theta_2 \in [\theta]_=) \\ &\text{ iff } t \sim_\gamma t' \ . \end{aligned}$$

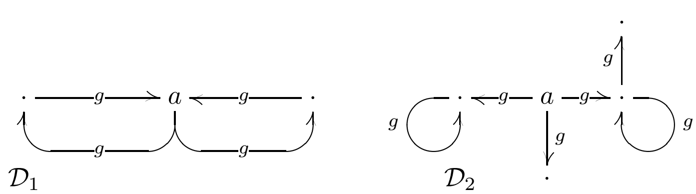
We continue by showing that the mapping is one-to-one. Assume $[\theta_1]_=, [\theta_2]_= \in \mathcal{R}(\mathcal{D}/{\sim})/{=}$ are distinct. Then, without loss of generality there are t, t' such that $t \neq_{\theta_1} t'$ and $t =_{\theta_2} t'$. Then there exists a context $C'[\ ]$ such that $\theta_1(C'[t]) \neq \theta_1(C'[t'])$ while $\theta_2(C[t]) = \theta_2(C[t'])$ for all contexts $C[\ ]$. Let η and γ be runs extracted from θ_1 and θ_2 respectively. Recall that if η_θ is extracted from θ, then $\eta_\theta(t)$ is an element of $\theta(t) \in D/{\sim}$ for any term t. Then $C'[t] \not\sim_\eta C'[t']$ and $C[t] \sim_\gamma C[t']$ for all contexts $C[\ ]$, *i.e.*, $[\theta_1]_=$ and $[\theta_2]_=$ are mapped to distinct elements in $\mathcal{R}(\mathcal{D})/{\sim}$.

We conclude by showing that the mapping is onto. Choose $[\eta]_\sim \in \mathcal{R}(\mathcal{D})/{\sim}$. Define $\theta : T_\Sigma \longrightarrow D/{\sim}$ by $\theta(t) = [\eta(t)]_\sim$. By an inductive argument one can show that $\theta \in \mathcal{R}(\mathcal{D}/{\sim})$. Let γ be a run extracted from θ. Then for any term t, $\gamma(t) \in \theta(t)$, *i.e.*, $\gamma(t) \sim \eta(t)$. So

$$\begin{aligned} t \sim_\gamma t' &\text{ iff } \forall C[\ ].\, \gamma(C[t]) \sim \gamma(C[t']) \\ &\text{ iff } \forall C[\ ].\, \eta(C[t]) \sim \eta(C[t']) \\ &\text{ iff } t \sim_\eta t' \ , \end{aligned}$$

i.e., $[\gamma]_\sim = [\eta]_\sim$, hence $[\theta]_=$ is mapped to $[\eta]_\sim$. ∎

Notice that taking the "quotients" on the spaces $\mathcal{R}(\mathcal{D}/{\sim})$ and $\mathcal{R}(\mathcal{D})$ is necessary, as one of the following examples shows, where a is a constant and g a unary symbol.



Let $S \not\leftrightarrow S'$ denote that there exists no bijection between the sets S and S'. Then there exist $\mathcal{D}$-congruences $\sim$ such that

$$\begin{array}{lll} \mathcal{R}(\mathcal{D}_1) \not\leftrightarrow \mathcal{R}(\mathcal{D}_1/{\sim}) & \mathcal{R}(\mathcal{D}_1) \not\leftrightarrow \mathcal{R}(\mathcal{D}_1/{\sim})/{=} & \mathcal{R}(\mathcal{D}_2/{\sim}) \not\leftrightarrow \mathcal{R}(\mathcal{D}_2/{\sim})/{=} \\ \mathcal{R}(\mathcal{D}_1) \not\leftrightarrow \mathcal{R}(\mathcal{D}_1)/{\sim} & \mathcal{R}(\mathcal{D}_2/{\sim}) \not\leftrightarrow \mathcal{R}(\mathcal{D}_2)/{\sim} & \\ \mathcal{R}(\mathcal{D}_2) \not\leftrightarrow \mathcal{R}(\mathcal{D}_2)/{=} & \mathcal{R}(\mathcal{D}_1/{\sim}) \not\leftrightarrow \mathcal{R}(\mathcal{D}_1)/{=} & \end{array}$$

The following proposition shows that if a rational map is a refinement, then this property is preserved by the quotient construction.

Proposition 24. *Given a refinement $\widehat{h} : \mathcal{R}(\mathcal{D}) \longrightarrow \mathcal{R}(\mathcal{D}')$. Let $\sim \subseteq D \times D$ be defined by $d \sim d'$ if and only if $h(d) = h(d')$. Let $\approx$ be any $\mathcal{D}$-congruence in $\sim$. Then $g : D/{\approx} \longrightarrow D'$ defined by $[d]_\approx \mapsto h(d)$ is well-defined and defines a refinement $\widehat{g} : \mathcal{R}(\mathcal{D}/{\approx}) \longrightarrow \mathcal{R}(\mathcal{D}')$.*

Proof. Clearly, since $\approx \subseteq \sim$, g is well-defined. Given $f \in \Sigma_n$, $[d_1]_\approx, \ldots, [d_n]_\approx \in D/{\approx}$. Then

$$\begin{aligned} g(E/{\approx}_f([d_1]_\approx, \ldots, [d_n]_\approx)) &= g(\{[d]_\approx \mid d \in E_f(d_1, \ldots, d_n)\}) \\ &= h(\{d \mid d \in E_f(d_1, \ldots, d_n)\}) \\ &= h(E_f(d_1, \ldots, d_n)) \\ &\subseteq E'_f(h(d_1), \ldots, h(d_n)) \\ &= E'_f(g([d_1]_\approx), \ldots, g([d_n]_\approx)) \ . \end{aligned}$$

Hence, g defines a rational map. Given $\theta \in \mathcal{R}(\mathcal{D}/{\approx})$. Let η_θ be a run extracted from θ. Then $\eta_\theta(t) \in \theta(t)$, so

$$(\widehat{g}(\theta))(t) = g(\theta(t)) = h(\eta_\theta(t)) = (\widehat{h}(\eta_\theta))(t) \ ,$$

i.e., $\widehat{g}(\theta) = \widehat{h}(\eta_\theta)$. Hence, $\widehat{g}$ must be one-to-one.

Pick $\theta' \in \mathcal{R}(\mathcal{D}')$. Then there exists an $\eta \in \mathcal{R}(\mathcal{D})$ such that $\widehat{h}(\eta) = \theta'$. The map $\theta : T_\Sigma \longrightarrow D/{\approx}$, defined by $\theta(t) = [\eta(t)]_\approx$ is a run in $\mathcal{R}(\mathcal{D}/{\approx})$, from which η can be extracted. Hence, $\widehat{g}(\theta) = \widehat{h}(\eta) = \theta'$, and $\widehat{g}$ is onto. ∎

5 Complexity of Rational Embeddings

In [Koz95] it is shown that for a finite set of variables X, one can define finitary rational spaces $\mathcal{R}(2^F, \mathcal{S})$, given a finite system of set constraints $\mathcal{S}$ and a subset of set term F satisfying certain closure properties. Moreover, it is shown that up to logical equivalence, the finite systems of set constraints over X correspond to the finitary rational subspaces—induced by the spaces of the form $\mathcal{R}(2^F, \mathcal{S})$—of $\mathcal{R}(2^X, \emptyset)$, and that this correspondence preserves the partial order of logical entailment between finite systems of set constraints over X and so-called X-preserving [2] rational embeddings between spaces $\mathcal{R}(2^F, \mathcal{S})$.

[2] Deciding if a map $h : \mathcal{D} \longrightarrow \mathcal{D}'$ is X-preserving can be done in polynomial time.

In this section we examine the complexity of some decision problems related to rational embeddings. When stating the problem instances in terms of rational spaces $\mathcal{R}(\mathcal{D})$ and rational maps $\widehat{h}$, we implicitly assume that they are stated in terms of the underlying $\mathcal{D}$ and h.

First, some preliminary results are necessary.

Definition 25. Given $\mathcal{D}$. The *non-emptiness problem* is the problem of deciding whether or not $\mathcal{R}(\mathcal{D}) \neq \emptyset$. □

Lemma 26. *Given a finite $\mathcal{D}$. The problem of deciding if $\mathcal{R}(\mathcal{D}) \neq \emptyset$ is NP-complete.*

Proof. To show NP-hardness, we reduce the NP-complete 3-CNF satisfiability problem [HU79] to the accessibility problem. We assume the reader is familiar with 3-CNF satisfiability.

Let $F = F_1 \wedge \cdots \wedge F_m$ be an expression in 3-CNF, where each F_i, $1 \leq i \leq m$, is a clause of the form $(\alpha_{i_1} \vee \alpha_{i_2} \vee \alpha_{i_3})$ and each α_{i_j}, $1 \leq j \leq 3$, is a literal, *i.e.*, a negated or non-negated boolean variable. Let *Var* be the set $\{x_1, \ldots, x_k\}$ of boolean variables that occur in F. Let $\mathit{Var}(\alpha_{i_j}) = \ell$, where x_ℓ is the variable in *Var* that occurs in α_{i_j}.

Define $\Sigma = \Sigma_0 \cup \Sigma_3$, where

$$\begin{aligned} \Sigma_0 &= \{a_1, \ldots, a_k\} \\ \Sigma_3 &= \{f_1, \ldots, f_m\} \ . \end{aligned}$$

Define a Σ-hypergraph $\mathcal{D} = (D, E)$ by

$$\begin{aligned} D &= \{tt_1, ff_1, \ldots, tt_k, ff_k, *\} \\ E_{a_\ell} &= \{tt_\ell, ff_\ell\} \ , \quad 1 \leq \ell \leq k \ . \end{aligned}$$

In order to define E_{f_i}, let us say that $d \in D$ *matches* α_{i_j} if $d \in \{tt_\ell, ff_\ell\}$, where $\ell = \mathit{Var}(\alpha_{i_j})$. Intuitively, d then corresponds to a truth assignment of α_{i_j} by interpreting $d = tt_\ell$ as x_ℓ being assigned the value true and $d = ff_\ell$ as x_ℓ being assigned the value false. Assume d_1, d_2, and d_3 match $\alpha_{i_1}, \alpha_{i_2}$, and α_{i_3}, respectively, and the corresponding truth assignments are consistent, *i.e.*, if $d_{j_1}, d_{j_2} \in \{tt_\ell, ff_\ell\}$, $1 \leq j_1, j_2 \leq 3$, then $d_{j_1} = d_{j_2}$. Then d_1, d_2, and d_3 are said to *falsify* F_i, if F_i evaluates to false under the corresponding truth assignments. Now define

$$E_{f_i}(d_1, d_2, d_3) = \begin{cases} \emptyset & \text{, if } d_1, d_2, \text{ and } d_3 \text{ falsify } F_i \\ \{*\} & \text{, else} \end{cases}$$

$\mathcal{D}$ can be build in time polynomial in the size of F and has the property that $\mathcal{R}(\mathcal{D}) \neq \emptyset$ if and only if F is satisfiable. An inductive argument shows that a satisfying truth assignment $\sigma : \mathit{Var} \longrightarrow \{\text{true,false}\}$ uniquely determines a run $\theta : T_\Sigma \longrightarrow D$ that maps a_ℓ to tt_ℓ or ff_ℓ depending on whether $\sigma(x_\ell)$ is true or false, respectively. Also, from any run θ of $\mathcal{D}$, a satisfying truth assignment can be obtained by defining $\sigma(x_\ell)$ to be true or false depending on whether $\theta(a_\ell)$ is tt_ℓ or $f\!f_\ell$, respectively.

To see that the problem of determining whether not $\mathcal{R}(\mathcal{D}) \neq \emptyset$ lies in NP, just observe that a finite hypergraph $\mathcal{D}$ has a run if and only if it has an entire subhypergraph. ■

Definition 27. Given $\mathcal{D}$ and $d \in D$. d is said to be $\mathcal{D}$-*accessible* if there exists a run $\theta \in \mathcal{R}(\mathcal{D})$ and a ground term t such that $\theta(t) = d$. The *accessibility problem* is the problem of deciding whether or not d is $\mathcal{D}$-accessible. □

Lemma 28. *Given a finite $\mathcal{D}$ and a $d \in D$. The accessibility problem is NP-hard.*

Proof. This follows from an easy modification of the proof of Lemma 26. ■

Lemma 29. *Given a finite $\mathcal{D}$ and a $d \in D$. The accessibility problem lies in NP.*

Proof. From [AKW95] it follows that the problem can be reduced in time polynomial in the size of $\mathcal{D}$ to the Nonlinear Reachability Problem (NRP). Since the NRP is NP-complete [Ste94b] it follows that the accessibility problem is in NP. ■

The following lemma provides a useful characterization of rational embeddings.

Lemma 30. *Given a rational map $\widehat{h} : \mathcal{R}(\mathcal{D}) \longrightarrow \mathcal{R}(\mathcal{D}')$ between finitary rational spaces. Then, $\widehat{h}$ is a rational embedding if and only if there exists no Σ-hypergraph $\mathcal{D}^* = (D^*, E^*)$ such that*

$$\begin{array}{rl} (i) & D^* \subseteq D \times D \text{ and } \forall (d,d') \in D^*.\, h(d) = h(d') \\ (ii) & (d,d') \in E^*_f((d_1,d'_1),\ldots,(d_n,d'_n)) \;\Rightarrow \\ & \qquad (d \in E_f(d_1,\ldots,d_n) \wedge\; d' \in E_f(d'_1,\ldots,d'_n)) \\ (iii) & \exists (d_0,d'_0) \in D^*.\, d_0 \neq d'_0 \;\wedge\; (d_0,d'_0) \text{ is } \mathcal{D}^*\text{-accessible.} \end{array}$$

Proof. Assume $\widehat{h}$ is not a rational embedding. Choose $\eta_1, \eta_2 \in \mathcal{R}(\mathcal{D})$ such that $\widehat{h}(\eta_1) = \widehat{h}(\eta_2)$ and $\eta_1 \neq \eta_2$. Let t_0 be a ground term such that $\eta_1(t_0) \neq \eta_2(t_0)$. Let

$$D^* \;=\; \{(\eta_1(t), \eta_2(t)) \,|\, t \in T_\Sigma\} \;.$$

For $f \in \Sigma_n$ and $(d_1,d'_1),\ldots,(d_n,d'_n) \in D^*$, define

$$\begin{aligned} E^*_f((d_1,d'_1),\ldots,(d_n,d'_n)) = \{(d,d') \,|\, &\exists f(t_1,\ldots,t_n) \in T_\Sigma. \\ &(\eta_1(t_1),\eta_2(t_1)) = (d_1,d'_1),\ldots, \\ &(\eta_1(t_n),\eta_2(t_n)) = (d_n,d'_n) \;\wedge \\ &(\eta_1(f(t_1,\ldots,t_n)),\eta_2(f(t_1,\ldots,t_n))) = (d,d')\} \;. \end{aligned}$$

Then (i) holds because $\widehat{h}(\eta_1) = \widehat{h}(\eta_2)$ and (ii) because η_1 and η_2 are runs over $\mathcal{D}$. Also, (iii) holds because $\theta : T_\Sigma \longrightarrow D^*$ defined by $\theta(t) = (\eta_1(t), \eta_2(t))$ is a run of $\mathcal{D}^*$ and $\theta(t_0) = (d_0, d'_0)$, where $d_0 \neq d'_0$.

Conversely, assume (i)–(iii) are true. Let θ be a run witnessing that (d_0, d'_0) is $\mathcal{D}^*$-accessible. Define $\eta_1 : T_\Sigma \longrightarrow D$ and $\eta_2 : T_\Sigma \longrightarrow D$ by $\eta_1(t) = d$ and $\eta_2(t) = d'$, where $\theta(t) = (d, d')$. By (ii) η_1 and η_2 are runs of $\mathcal{D}$, by (i) $\widehat{h}(\eta_1) = \widehat{h}(\eta_2)$, and by (iii) $\eta_1 \neq \eta_2$. Hence, $\widehat{h}$ is not a rational embedding. ■

Our first result on rational embeddings is:

Theorem 31. *Given a rational map $\widehat{h} : \mathcal{R}(\mathcal{D}) \longrightarrow \mathcal{R}(\mathcal{D}')$ between finitary rational spaces. The problem of deciding whether or not $\widehat{h}$ is a rational embedding is co-NP-complete.*

Proof. To show the hardness, we reduce the complement of the non-emptiness problem to this problem. Given $\mathcal{D}'$, let $\mathcal{D}$ denote the disjoint union of two copies of $\mathcal{D}'$, and let $\widehat{h} : D \longrightarrow D'$ denote the function which maps a state in D to the state in D' of which it is a copy. $\widehat{h}$ is a rational embedding if and only if $\mathcal{R}(\mathcal{D}) = \emptyset$.

From Lemma 30 it follows that the problem lies in co-NP, because the problem of determining the existence of a $\mathcal{D}^*$ as defined in the proof of Lemma 30 can be shown to be in NP using Lemma 29. ■

As an example of an application of Theorem 31, we obtain the following corollary.

Corollary 32. *Given a finitary rational space $\mathcal{R}(\mathcal{D})$. The problem of deciding whether or not $|\mathcal{R}(\mathcal{D})| > 1$ is NP-complete.*

Proof. The mapping $h : D \longrightarrow \{*\}$ defines a rational map $\widehat{h} : \mathcal{R}(\mathcal{D}) \longrightarrow \mathcal{R}(\mathcal{E})$, where $\mathcal{E}$ is the entire Σ-hypergraph over $\{*\}$, that is not a rational embedding if and only if $|\mathcal{R}(\mathcal{D})| > 1$. ■

Our second result on rational embeddings is:

Theorem 33. *Given two finitary rational spaces $\mathcal{R}(\mathcal{D})$ and $\mathcal{R}(\mathcal{D}')$. The problem of deciding whether or not there exists a rational embedding $\widehat{h} : \mathcal{R}(\mathcal{D}) \longrightarrow \mathcal{R}(\mathcal{D}')$ is NP-hard, co-NP-hard, and lies in Σ_2^P.*

Proof. The co-NP-hardness result can be obtained by combining the techniques from Theorem 31 and Corollary 32.

The NP-hardness result can be obtained by a reduction from the satisfiability problem for 3-CNF. Given F, a 3-CNF, one constructs $\mathcal{D}$ and $\mathcal{D}'$, such that $\mathcal{R}(\mathcal{D})$ is a singleton and such that there exists a rational map $\widehat{h} : \mathcal{R}(\mathcal{D}) \longrightarrow \mathcal{R}(\mathcal{D}')$ if and only if F is satisfiable. The constructions resembles that in the proof of Lemma 26. A map $h : D \longrightarrow D'$ will correspond to a truth assignment to *Var* satisfying F.

To see that the problems lies in Σ_2^P, observe that it can be formulated as

$$\{s_{(\mathcal{D},\mathcal{D}')} \mid \exists s_h . \forall s_{\mathcal{D}^*}, s_{(d_0,d'_0)}, s_{\mathit{comp}(\mathcal{D}^*,(d_0,d'_0))} . \\ (s_h, s_{\mathcal{D}^*}, s_{(d_0,d'_0)}, s_{\mathit{comp}(\mathcal{D}^*,(d_0,d'_0))}, s_{(\mathcal{D},\mathcal{D}')}) \in L\} \ ,$$

where $|s_h| < p(|s_{(\mathcal{D},\mathcal{D}')}|)$ and $|s_{\mathcal{D}^*} s_{(d_0,d'_0)} s_{comp(\mathcal{D}^*,(d_0,d'_0))}| < p(|s_{(\mathcal{D},\mathcal{D}')}|)$ for some polynomial p, and $L \in P$ is the language of a deterministic polynomial time Turing machine that checks that if the string

- $s_{(\mathcal{D},\mathcal{D}')}$ encodes two hypergraphs $\mathcal{D}$ and $\mathcal{D}'$,
- s_h encodes a mapping $h : D \longrightarrow D'$ satisfying (6),
- $s_{\mathcal{D}^*}$ encodes a hypergraph as described in the proof of Lemma 30, given $\mathcal{D}$, $\mathcal{D}'$, and h, and
- $s_{(d_0,d'_0)}$ encodes a pair (d_0, d'_0) belonging to $\mathcal{D}^*$ such that $d_0 \neq d'_0$,

then $s_{comp(\mathcal{D}^*,(d_0,d'_0))}$ is not an encoding of an accepting computation on input $(s_{\mathcal{D}^*}, s_{(d_0,d'_0)})$ of a (given) nondeterministic polynomial time Turing machine solving the accessibility problem, given the encoding $(s_{\mathcal{D}^*}, s_{(d_0,d'_0)})$. ■

6 Conclusion

In this paper we continued the investigation of the rational spaces introduced in [Koz95]. We gave a Myhill-Nerode-like characterization of rational points and results that suggest that rational points in an essential way captures the topological structure of finitary rational spaces. Furthermore, congruences on Σ-hypergraphs were investigated as well as complexity issues for rational maps.

As for future work, one might try to determine the complexity of deciding if a given rational map is a refinement, or whether or not two finitary rational spaces are rationally equivalent. Rational equivalence, which is defined i terms of spans of refinments, could possibly be recasted using Joyal, Nielsen, and Winskel's theory of open maps [JNW93].

Also, one could try to close the gap in Theorem 33.

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