On the Myhill-Nerode Theorem for Trees[*]

Dexter Kozen[†]
Cornell University
kozen@cs.cornell.edu

The *Myhill-Nerode Theorem* as stated in [6] says that for a set R of strings over a finite alphabet Σ, the following statements are equivalent:

(i) R is regular

(ii) R is a union of classes of a right-invariant equivalence relation of finite index

(iii) the relation $\equiv_R$ is of finite index, where $x \equiv_R y$ iff $\forall z \in \Sigma^* \; xz \in R \leftrightarrow yz \in R$.

This result generalizes in a straightforward way to automata on finite trees. I rediscovered this generalization in connection with work on finitely presented algebras, and stated it without proof or attribution in [7, 8], being at that time under the impression that it was folklore and completely elementary. It was again rediscovered independently by Z. Fülöp and S. Vágvölgyi and reported in a recent contribution to this Bulletin [5]. In that paper they attribute the result to me.

In fact, the result goes back at least ten years earlier to the late 1960s. It is difficult to attribute it to any one paper, since it seems to have been in the air at a time when the theory of finite automata on trees was undergoing intense development. In a sense, it is an inevitable consequence Myhill and Nerode's work [9, 10], since "conventional finite automata theory goes through for the generalization—and it goes through quite neatly" [11]. The first explicit

[*] *Bull. Europ. Assoc. Theor. Comput. Sci.* 47 (June 1992), 170–173.

[†] Computer Science Department, Cornell University, Ithaca, New York 14853, USA

mention of the equivalence of the tree analogs of (i) and (ii) seems to be by Brainerd [2, 3] and Eilenberg and Wright [4], although the latter claim that their paper "contains nothing that is essentially new, except perhaps for a point of view" [4]. A relation on trees analogous to $\equiv_R$ was defined and clause (iii) added explicitly by Arbib and Give'on [1, Definition 2.13], although it is also essentially implicit in work of Brainerd [2, 3].

All the cited papers from the 1960s involve heavy use of universal algebra and/or category theory. In these papers, a tree automaton is a finite Σ-algebra, and the map $\widehat{\delta}$ (see below) is a Σ-algebra homomorphism. Although exceedingly elegant, this approach renders the result less accessible to the average computer science undergraduate. Fülöp and Vágvölgyi take a somewhat different approach, appealing to the theory of term rewriting systems, tree transducers, and NTS grammars. Again, although this approach reveals some interesting and fundamental connections, it is rather involved and not suitable fare for undergraduates.

In contrast, the proof I had in mind when writing [7, 8] is a straightforward and comparatively mundane generalization of [6]. It can be appreciated by computer science undergraduates familiar with the Myhill-Nerode Theorem but with no knowledge of universal algebra, category theory, or term rewriting systems.

My purposes in writing this note are threefold: to set the record straight with respect to attribution; to apologize to Fülöp and Vágvölgyi for giving them the impression that I should be credited with the result; and to present an elementary proof in the style of [6].

Definitions

Let Σ be a finite ranked alphabet. The rank of $f \in \Sigma$ is called its *arity.* The set of n-ary elements of Σ is denoted Σ_n. The set of ground terms over Σ is denoted T_Σ. A *congruence* on T_Σ is an equivalence relation $\equiv$ such that $fs_1 \ldots s_n \equiv ft_1 \ldots t_n$ whenever $f \in \Sigma_n$ and $s_i \equiv t_i$, $1 \leq i \leq n$. A congruence $\equiv$ is *finitely generated* if it is generated by a finite subrelation. It is of *finite index* if there are only finitely many $\equiv$-classes. It *respects* $A \subseteq T_\Sigma$ if A is a union of $\equiv$-classes.

A *(deterministic, bottom-up) tree automaton* over Σ is a tuple

$$M = (Q, \Sigma, F, \delta)$$

where Q is a set of *states*, $F \subseteq Q$ is a set of *final states*, and δ is a *transition function*

$$\delta : \bigcup_n \Sigma_n \times Q^n \quad \rightarrow \quad Q \ .$$

In other words, δ takes an input symbol $f \in \Sigma$ and an n-tuple of states $q_1, \ldots, q_n$, where n is the arity of f, and produces a next state $\delta(f, q_1, \ldots, q_n) \in Q$.

Tree automata over Σ run on ground terms over Σ. Informally, an automaton starts at the leaves and moves upward, associating a state with each subterm inductively. If the immediate subterms $t_1, \ldots, t_n$ of the term $ft_1 \ldots t_n$ are labeled with states $q_1, \ldots, q_n$ respectively, then the term $ft_1 \ldots t_n$ will be labeled with state $\delta(f, q_1, \ldots, q_n)$. The term is accepted if the state labeling the root is in F.

Formally, define the labeling function $\widehat{\delta} : T_\Sigma \rightarrow Q$ inductively by

$$\widehat{\delta}(ft_1 \ldots t_n) \quad = \quad \delta(f, \widehat{\delta}(t_1), \ldots, \widehat{\delta}(t_n)) \ .$$

Note that the basis of the induction is included in this definition: $\widehat{\delta}(c) = \delta(c)$ for c nullary.

The term t is said to be *accepted* by M if $\widehat{\delta}(t) \in F$. The set of terms accepted by M is denoted $\mathcal{L}(M)$. A set of terms is called *regular* if it is $\mathcal{L}(M)$ for some M.

This definition extends the usual definition of automata on finite strings in a natural way: we can think of an automaton on strings over a finite alphabet Σ as a tree automaton over $\Sigma \cup \{\Box\}$ turned on its side, where we assign $\Box$ arity 0 and elements of Σ arity 1.

The Myhill-Nerode Theorem for Trees

For a given $R \subseteq T_\Sigma$, define $s \equiv_R t$ if for all terms u with exactly one occurrence of a variable x and no other variables,

$$u[x/s] \in R \quad \leftrightarrow \quad u[x/t] \in R \ .$$

Theorem (Myhill-Nerode Theorem for trees) *Let $R \subseteq T_\Sigma$. The following are equivalent:*

(i) R is regular

(ii) there exists a finitely generated congruence of finite index respecting R

(iii) the relation $\equiv_R$ is of finite index.

Proof. (i) → (iii) Suppose $R = \mathcal{L}(M)$ where $M = (Q, \Sigma, F, \delta)$. We show that if $\widehat{\delta}(s) = \widehat{\delta}(t)$ then $s \equiv_R t$, thus there are no more $\equiv$-classes than states of M. If $\widehat{\delta}(s) = \widehat{\delta}(t)$ and u is any term with exactly one occurrence of a variable x, then according to the behavior of the machine,

$$\widehat{\delta}(u[x/s]) \quad = \quad \widehat{\delta}(u[x/t]) \ .$$

This follows formally from an easy inductive argument on the depth of u. Thus

$$u[x/s] \in R \quad \leftrightarrow \quad \widehat{\delta}(u[x/s]) \in F \quad \leftrightarrow \quad \widehat{\delta}(u[x/t]) \in F \quad \leftrightarrow \quad u[x/t] \in R \ .$$

Since u was arbitrary, $s \equiv_R t$.

(iii) → (ii) We show that $\equiv_R$ is a finitely generated congruence respecting R. It is clearly an equivalence relation. It is also a congruence, since if f is n-ary and $s_i \equiv_R t_i$, $1 \le i \le n$, then for any u with exactly one occurrence of a variable x,

$$\begin{aligned} u[x/fs_1 \ldots s_{i-1}s_it_{i+1} \ldots t_n] \in R \quad &\leftrightarrow \quad u[x/fs_1 \ldots s_{i-1}yt_{i+1} \ldots t_n][y/s_i] \in R \\ &\leftrightarrow \quad u[x/fs_1 \ldots s_{i-1}yt_{i+1} \ldots t_n][y/t_i] \in R \\ &\leftrightarrow \quad u[x/fs_1 \ldots s_{i-1}t_it_{i+1} \ldots t_n] \in R \ , \end{aligned}$$

therefore

$$fs_1 \ldots s_{i-1}s_it_{i+1} \ldots t_n \quad \equiv_R \quad fs_1 \ldots s_{i-1}t_it_{i+1} \ldots t_n \ ,$$

and $fs_1 \ldots s_n \equiv_R ft_1 \ldots t_n$ follows from transitivity. It respects R, since if $s \equiv_R t$, then

$$s \in R \quad \leftrightarrow \quad x[x/s] \in R \quad \leftrightarrow \quad x[x/t] \in R \quad \leftrightarrow \quad t \in R \ .$$

It is finitely generated, since any congruence $\equiv$ of finite index is: if $U \subseteq T_\Sigma$ is a complete set of representatives for the $\equiv$-classes, then $\equiv$ is generated by the finite subrelation consisting of all equations in $\equiv$ of the form

$$fu_1 \ldots u_n \quad \equiv \quad u$$

for $u_1, \ldots, u_n, u \in U$ and $f \in \Sigma_n$. This is because every term is equivalent to some $u \in U$ in the congruence generated by this subrelation, as an easy inductive argument shows.

(ii) → (i) Let $\equiv$ be the congruence, and let $[t]$ denote the $\equiv$-class of t. Form an automaton M with states $Q = \{[t] \mid t \in T_\Sigma\}$, final states $F = \{[t] \mid t \in R\}$, and transition function

$$\delta(f, [t_1], \ldots, [t_n]) \quad = \quad [ft_1 \ldots t_n] \; .$$

The function δ is well-defined, since if $[s_i] = [t_i]$, $1 \leq i \leq n$, then $[fs_1 \ldots s_n] = [ft_1 \ldots t_n]$. Moreover, an easy induction shows that $\widehat{\delta}(t) = [t]$ for all t, thus

$$t \in R \quad \leftrightarrow \quad [t] \in F \quad \leftrightarrow \quad \widehat{\delta}(t) \in F \quad \leftrightarrow \quad t \in \mathcal{L}(M) \; .$$

□

In complete analogy with the case of strings, the congruences on T_Σ of finite index respecting R are in one-to-one correspondence (up to isomorphism) with the deterministic bottom-up finite tree automata with no inaccessible states accepting R, and there is a unique minimal such automaton corresponding to the congruence $\equiv_R$.

Acknowledgements

This research was done while the author was on sabbatical at Aarhus University, Denmark. Support from the Danish Research Academy, the National Science Foundation under grant CCR-8806096, the John Simon Guggenheim Foundation, and the U.S. Army Research Office through the ACSyAM branch of the Mathematical Sciences Institute of Cornell University, contract DAAL03-91-C-0027 is gratefully acknowledged.

References

[1] M. A. Arbib and Y. Give'on. Algebra automata I: parallel programming as a prolegomenon to the categorical approach. *Infor. and Control*, 12:331–345, 1968.

[2] W. S. Brainerd. *Tree Generating Systems and Tree Automata.* PhD thesis, Purdue University, 1967.

[3] W. S. Brainerd. The minimalization of tree automata. *Infor. and Control*, 13:484–491, 1968.

[4] S. Eilenberg and J. B. Wright. Automata in general algebra. *Infor. and Control*, 11:452–470, 1967.

[5] Z. Fülöp and S. Vágvölgyi. Congruential tree languages are the same as recognizable tree languages—a proof for a theorem of D. Kozen. *Bull. Europ. Assoc. Theor. Comput. Sci.*, 39:175–184, October 1989.

[6] J. E. Hopcroft and J. D. Ullman. *Introduction to Automata Theory, Languages, and Computation.* Addison-Wesley, 1979.

[7] D. Kozen. *Complexity of Finitely Presented Algebras.* PhD thesis, Cornell University, May 1977.

[8] D. Kozen. Complexity of finitely presented algebras. In *Proc. 9th ACM Symp. Theory of Comput.*, pages 164–177, May 1977.

[9] J. Myhill. Finite automata and the representation of events. Technical Report 57-264, WADC, 1957.

[10] A. Nerode. Linear automata transformations. *Proc. Amer. Math. Soc.*, 9:541–544, 1958.

[11] J. W. Thatcher and J. B. Wright. Generalized finite automata theory with an application to a decision problem of second order logic. *Math. Syst. Theory*, 2:57–81, 1968.