Definability with Bounded Number of Bound Variables

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Abstract

A theory satisfies the *k-variable property* if every first-order formula is equivalent to a formula with at most k bound variables (possibly reused). Gabbay has shown that a model of temporal logic satisfies the k-variable property for some k if and only if there exists a finite basis for the temporal connectives over that model. We give a model-theoretic method for establishing the k-variable property, involving a restricted Ehrenfeucht-Fraisse game in which each player has only k pebbles. We use the method to unify and simplify results in the literature for linear orders. We also establish new k-variable properties for various theories of bounded-degree trees, and in each case obtain tight upper and lower bounds on k. This gives the first finite basis theorems for branching-time models of temporal logic.

1 Introduction

A first-order theory Σ satisfies the *k-variable property* if every first-order formula is equivalent under Σ to a formula with at most k bound variables (possibly reused). For example, in an arbitrary partial order, five bound variables are needed to express the statement "there are at least five elements below x," but in a linear order, two variables suffice:

$$\exists y \; y < x \wedge (\exists x \; x < y \wedge (\exists y \; y < x \wedge (\exists x \; x < y \wedge (\exists y \; y < x)))) \; . \qquad (1)$$

The k-variable property is important in temporal logic. Gabbay [7] has shown that a model of temporal logic satisfies the k-variable property for some k if and only if there exists a finite basis for the first-order-expressible temporal connectives over that model, in the same sense that $\vee$ and $\neg$ form a basis for the propositional connectives.

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Kamp [11] showed that any Dedekind-complete linear order with arbitrary monadic predicates admits a finite basis for the temporal connectives. This result was extended to other linear time structures by Stavi [16]. Amir and Gabbay [1] showed that any definable lexicographic product of time structures admitting a finite basis also admits a finite basis. This result gave the first infinite non-linear structures admitting a finite basis, although up to now no results have been established for trees.

The methods used by these researchers were largely syntactic. In this paper we give a model-theoretic method for establishing the k-variable property uniformly for all models of certain first-order theories. The method uses a variant of the Ehrenfeucht-Fraisse game [3, 5] which allows each player only k pebbles [9, 14, 10].

Applying this method to the theory of linear order, we are able to unify the results of [11, 16]. We also establish new k-variable expressiveness results for various theories of bounded-degree trees, and in each case obtain tight upper and lower bounds on k. Using Gabbay's result [7], these results imply the existence of a finite basis for the first-order-expressible temporal connectives over tree models of bounded degree.

2 A Model-Theoretic Lemma

Let L be a first-order language with individual variables $x_1, x_2, \ldots$. A *partial valuation* over a structure A for L is a partial function $u : \{x_1, x_2, \ldots\} \to A$. The domain of u is denoted ∂u. The cardinality of ∂u is denoted $|u|$. A *(k-)configuration* over A, B is a pair (u, v), where u is a partial valuation over A and v is a partial valuation over B, such that $\partial u = \partial v \subseteq \{x_1, \ldots, x_k\}$. If $L' \subseteq L$, an *L'-type* in the variables $x_1, \ldots, x_k$ is a maximal consistent set of L' formulas all of whose free variables are among $x_1, \ldots, x_k$. If (u, v) is a k-configuration, then u and v are said to be *L'-equivalent* if they have the same L'-type; i.e., if for all formulas $\varphi \in L'$ with free variables in $\partial u = \partial v$,

$$A, u \models \varphi \quad \text{iff} \quad B, v \models \varphi \ .$$

Lemma 1 *Let Σ be a set of sentences in L. Let $L', L'' \subseteq L$ such that L' is closed under the propositional operators. The following two conditions are equivalent:*

(i) for all models A, B of Σ and k-configurations (u, v) over A, B, if u and v are L'-equivalent, then they are L''-equivalent;

(ii) for all $\varphi \in L''$ with free variables among $x_1, \ldots, x_k$, there exists a $\psi \in L'$ such that $\Sigma \models \varphi \leftrightarrow \psi$.

Remark Informally, condition (i) means that if u and v can be distinguished by a formula of L'', then they can be distinguished by a formula of L'. Thus, the lemma says intuitively that L'' has no more power than L' to distinguish such u and v if and only if L' subsumes L'' in expressive power, at least on formulas involving only free variables $x_1, \ldots, x_k$.

Proof. (ii) $\rightarrow$ (i) is immediate.

(i) $\rightarrow$ (ii): If $\Sigma \cup \{\varphi\}$ is inconsistent, take $\psi = \mathit{false}$ and we are done. Otherwise, let Γ be an arbitrary complete L'-type in the variables $x_1, \ldots, x_k$ consistent with $\Sigma \cup \{\varphi\}$. Then $\Sigma \cup \Gamma \cup \{\neg\varphi\}$ is inconsistent, otherwise models A, u and B, v of $\Sigma \cup \Gamma$ could be constructed with $\partial u = \partial v = \{x_1, \ldots, x_k\}$ such that $A, u \models \varphi$ and $B, v \models \neg\varphi$, violating (i). Therefore $\Sigma \cup \Gamma \models \varphi$. By compactness, there exists a $\psi_\Gamma \in \Gamma$ such that $\Sigma \models \psi_\Gamma \rightarrow \varphi$. Now φ is covered by all such ψ_Γ, in the sense that

$$\Sigma \quad \models \quad (\bigvee_{\Gamma} \psi_\Gamma) \leftrightarrow \varphi$$

where the infinitary join is taken over all L'-types Γ consistent with $\Sigma \cup \{\varphi\}$. Again by compactness, there is a finite set F of such Γ such that

$$\Sigma \quad \models \quad (\bigvee_{\Gamma \in F} \psi_\Gamma) \leftrightarrow \varphi \ ,$$

so we may take $\psi = \bigvee_{\Gamma \in F} \psi_\Gamma$. □

We are interested in a special case of the above lemma which applies to the k-variable property.

Definition 2 Define the *quantifier depth* of a formula φ inductively, as follows.

1. If φ is quantifier-free, then its quantifier depth is 0.
2. The quantifier depth of $\varphi \vee \psi$ or $\varphi \wedge \psi$ is the maximum of the quantifier depths of φ and ψ.
3. The quantifier depth of $\neg\varphi$ is the quantifier depth of φ.
4. The quantifier depth of $\forall x\ \varphi$ or $\exists x\ \varphi$ is one greater than the quantifier depth of φ.

□

For example, the quantifier depth of the formula (1) is 5.

Let $n, k \geq 0$. Define $L_{k,n}$ to be the sublanguage of L consisting of all formulas φ of quantifier depth at most n containing only variables $x_1, \ldots, x_k$. For example, the formula 1 is in $L_{2,5}$. Define

$$L_k \quad = \quad \bigcup_n L_{k,n} \ .$$

Thus $L = \bigcup_k L_k$.

Definition 3 A first-order theory Σ is said to satisfy the *k-variable property* if for all formulas $\varphi \in L$ with free variables among $x_1, \ldots, x_k$, there exists a $\psi \in L_k$ such that $\Sigma \models \varphi \leftrightarrow \psi$. □

In this special case, Lemma 1 gives

Corollary 4 *The following two conditions are equivalent:*

(i) for all models A, B of Σ and k-configurations (u, v) over A, B, if u and v are L_k-equivalent, then they are L-equivalent;

(ii) Σ satisfies the k-variable property.

3 An Ehrenfeucht-Fraisse Game with Bounded Number of Pebbles

Let Σ be a theory in a first-order language L with equality. Assume further that in every model of Σ, every finitely generated substructure is finite; i.e., the smallest substructure containing a given finite set is always finite. (This is a technical restriction that is used in the proofs below.) We have reduced the problem of establishing the k-variable property for Σ to checking the condition of Corollary 4(i). This will done using *Ehrenfeucht-Fraisse games* [3, 5].

Ehrenfeucht-Fraisse games have been used widely in theoretical computer science; see e.g. [4, 6, 8, 10, 12, 13, 17, 18]. Here we use a modified version in which the number of pebbles is finite [9, 14, 10].

Definition 5 Let A, B be structures for L and (u, v) a k-configuration. We call (u, v) a *local isomorphism* if the map $u(x) \mapsto v(x)$, $x \in \partial u$, is well-defined and extends to an isomorphism of the substructures of A and B generated by $\{u(x) \mid x \in \partial u\}$ and $\{v(x) \mid x \in \partial v\}$, respectively. That is, (u, v) is a local isomorphism if the relation

$$\{(t^{A,u}, t^{B,v}) \mid t \text{ is a term over } \partial u\} \quad \subseteq \quad A \times B$$

is a bijection and respects the functions and relations of L. □

Definition 6 (The game $G(u, v, k, n)$.) Let A, B be structures for L, $n, k \geq 0$, and (u, v) a k-configuration. The game $G(u, v, k, n)$ is played by two players, I and II, who take turns placing pebbles on elements of A and B. Player I tries to demonstrate that A and B are nonisomorphic, and Player II tries to make A and B appear isomorphic. There are $2k$ pebbles, each colored with one of k distinct colors $x_1, \ldots, x_k$, with exactly two pebbles of each color.

A configuration (u', v') denotes that the two pebbles colored x_i are currently occupying $u'(x_i) \in A$ and $v'(x_i) \in B$, for $x_i \in \partial u' = \partial v'$, and that the pebbles colored $x_i \notin \partial u'$ are not currently in play. The initial configuration is $(u_0, v_0) = (u, v)$. The players alternate, with Player I first. Each round consists of a move of Player I followed by a move of Player II. Player I can select any pebble and place it on an element of either A or B. Player II then has to place the other pebble of the same color on an element of the other structure. Play proceeds for n rounds, generating a sequence of configurations (u_t, v_t), $0 \le t \le n$. Player II wins the game if all the (u_t, v_t), $0 \le t \le n$, are local isomorphisms (Definition 5). Otherwise Player I wins. □

Definition 7 A *forced win* for Player II is defined by induction on n. Player II has a forced win in $G(u, v, k, 0)$ if (u, v) is a local isomorphism. Player II has a forced win in $G(u, v, k, n + 1)$ if (u, v) is a local isomorphism, and for all legal moves of Player I from configuration (u, v), there exists a legal move of Player II resulting in a configuration (u', v') such that Player II has a forced win in $G(u', v', k, n)$. Player I has a forced win if Player II does not. □

Intuitively, a player has a forced win if there is always a choice of moves for that player leading to a win, no matter how well his opponent plays.

Example 8 Consider the two-pebble game $G(\emptyset, \emptyset, 2, n)$ played on the linear orders $\mathcal{Z}$ and $\mathcal{Q}$. Player II has a forced win, as follows. In the first round, Player II plays anywhere in response to Player I's move. In the second round, if Player I plays in either structure to the left (right) of the pebble already on the board, then Player II does the same in the other structure. Subsequently, if Player I moves a pebble in either structure, Player II moves the corresponding pebble in the other structure so as to maintain the relative ordering of the pebbles in the two structures. Player II always wins, since every configuration is a local isomorphism.

On the other hand, Player I has a forced win in the three-pebble game $G(\emptyset, \emptyset, 3, 3)$, as follows. Player I starts by playing any point p in $\mathcal{Z}$. Player II responds by playing a point q in $\mathcal{Q}$. Now Player I plays $p + 1$ in $\mathcal{Z}$. Player II must play a point q' of $\mathcal{Q}$ to the right of q, otherwise Player I wins. Player I now plays any point of $\mathcal{Q}$ between q and q', and Player II is stuck. Note that Player I's winning strategy is based on the fact that $\mathcal{Q}$ and $\mathcal{Z}$ are distinguished by the property of *density*, which is expressible with three variables:

$$\forall x \forall z \, (x < z \rightarrow \exists y \; x < y < z) \, .$$

□

It is always to Player I's advantage to play a pebble not currently on the board, if possible, and to place a pebble on an element not currently covered by another pebble, if possible; from Player I's point of view, the more elements of A and B that are covered, the better. Any winning strategy for Player I that does not satisfy these conditions can be mapped into a winning strategy that does.

Lemma 9 *If Player II has a forced win in the game $G(u, v, k, n)$, then Player II has a forced win in the game $G(u', v', k', n')$, for any $n' \le n$, $k' \le k$, and u' and v' restrictions of u and v, respectively, to a smaller domain.*

We now prove a series of lemmas that will allow us to establish the relationship between the games $G(u, v, k, n)$ and the k-variable property. Lemmas 10 and 11 are technical. Lemma 12 is a generalization of [10], Theorem C.1, to structures allowing function symbols, provided that all finitely generated substructures are finite.

Lemma 10 *Let Σ be a first-order theory such that all finitely generated substructures of models of Σ are finite. Then there is a uniform bound on the size of substructures generated by k elements. That is, for all k there exists a bound b_k such that for any model A of Σ and substructure B of A generated by k elements, B contains no more than b_k elements.*

Proof. We use a compactness argument. Define the *depth* of a term inductively, as follows: constants and variables have depth 0, and a term of the form $f(t_1, \ldots, t_m)$ has depth $1 + \max\{\text{depth of } t_i \mid 1 \leq i \leq m\}$. Let D^k_m denote the set of terms of depth at most m over the variables $x_1, \ldots, x_k$. Then D^k_m is a finite set, although its size depends on the number of function symbols in L and their arity.

Let ρ_m be the formula

$$\bigwedge_{s \in D^k_{m+1}} \bigvee_{t \in D^k_m} s = t \ . \tag{2}$$

The formula ρ_m says that every element represented by a term of depth at most $m + 1$ over $x_1, \ldots, x_k$ is already represented by a term of depth at most m; in other words, every element of the substructure generated by $x_1, \ldots, x_k$ is represented by a term of depth at most m. Note that ρ_m is a quantifier-free formula of L over the variables $x_1, \ldots, x_k$, and that ρ_m logically implies ρ_{m+1}.

By the assumption that all finitely generated substructures of models of Σ are finite, we have

$$\Sigma \models \bigvee_{m=0}^{\infty} \rho_m \ .$$

By compactness, there is an n such that

$$\Sigma \models \rho_n \ .$$

We may therefore take $b_k = |D^k_n|$. □

We say that formulas φ and ψ are *equivalent under* Σ if $\Sigma \models \varphi \leftrightarrow \psi$.

Lemma 11 *Under the assumption of Lemma 10, there are only finitely many inequivalent formulas of $L_{k,n}$ under Σ.*

Proof. This lemma is similar to [15, Lemma 13.10, p. 251], except that we are in the presence of function symbols. By Lemma 10, there is a uniform bound on the size of substructures generated by k elements in any model of Σ. This is equivalent to the statement that there exists an $m = m_k$ such that $\Sigma \models \rho_m$, where ρ_m is the formula (2) defined in the proof of Lemma 10.

Consider the formula ρ_m. Using distributivity of $\wedge$ over $\vee$, rewrite ρ_m so that it is in disjunctive form

$$\bigvee_g \bigwedge_{t\in D^k_{m+1}} t = g(t) \ ,$$

where the outer join is over all maps

$$g : D^k_{m+1} \to D^k_m$$

assigning a term of depth at most m to each term of depth at most $m+1$. We extend each such g to domain $\bigcup_n D^k_n$ inductively, as follows: for $f(t_1,\ldots,t_d) \in D^k_{n+1} - D^k_n$, $n > m$, take

$$g(f(t_1,\ldots,t_d)) \quad = \quad g(f(g(t_1),\ldots,g(t_d))) \ .$$

This is well-defined, since all applications of g on the right hand side are to terms of smaller depth. By repeated application of the rule of substitution of equals for equals, we have that for all terms s over the variables $x_1,\ldots,x_k$,

$$\models (\bigwedge_{t\in D^k_{m+1}} t = g(t)) \to s = g(s) \ ;$$

moreover, for any atomic formula $R(s_1,\ldots,s_d)$, where R is a d-ary relation symbol and $s_1,\ldots,s_d$ are terms over the variables $x_1,\ldots,x_k$,

$$\models \ (\bigwedge_{t\in D^k_{m+1}} t = g(t)) \quad \to \quad (R(s_1,\ldots,s_d) \leftrightarrow R(g(s_1),\ldots,g(s_d))) \ .$$

From this and the fact that $\Sigma \models \rho_m$, we conclude that

$$\Sigma \ \models \ R(s_1,\ldots,s_d) \quad \leftrightarrow \quad \bigvee_g (\bigwedge_{t\in D^k_{m+1}} t = g(t) \wedge R(g(s_1),\ldots,g(s_d))) \ . \tag{3}$$

The right hand side of (3) is a quantifier-free formula containing only terms of depth at most $m+1$, and there are only finitely many such formulas up to propositional equivalence.

It follows immediately that $L_{k,0}$ contains only finitely many formulas up to equivalence under Σ. We next show by induction on n that the same is true for $L_{k,n}$. Assume this is true for $L_{k,r}$. Then $L_{k,r+1}$ consists of Boolean combinations of formulas φ and $\exists x_i\varphi$ for $\varphi \in L_{k,r}$ and $1 \le i \le k$. Up to equivalence, there are only finitely many of these. □

Lemma 12 *Let Σ be a first-order theory such that all finitely generated substructures of models of Σ are finite. Let A and B be models of Σ, and let (u,v) be a k-configuration. Then Player II has a forced win in the game $G(u,v,k,n)$ if and only if u and v are $L_{k,n}$-equivalent.*

Proof. We prove the lemma by induction on n. For the basis, Player II has a forced win in $G(u,v,k,0)$ iff (u,v) is a local isomorphism iff u and v agree on all quantifier-free formulas of L_k with variables among ∂u, i.e., u and v are $L_{k,0}$-equivalent. Now suppose $n > 0$.

($\rightarrow$) Suppose Player II has a forced win in the game $G(u, v, k, n)$. It suffices to show that u and v agree on all formulas of $L_{k,n}$ of the form $\exists x_i\ \psi$. Suppose $A, u \models \exists x_i\ \psi$. Let $a \in A$ such that $A, u[x_i/a] \models \psi$. If Player I should move the pebble colored x_i to a, then Player II has a response $b \in B$ such that Player II has a forced win in $G(u[x_i/a], v[x_i/b], k, n-1)$, by Definition 7. Since $\psi \in L_{k,n-1}$, by the induction hypothesis, $B, v[x_i/b] \models \psi$, thus $B, v \models \exists x_i\ \psi$. A symmetric argument shows that if $B, v \models \exists x_i\ \psi$ then $A, u \models \exists x_i\ \psi$.

($\leftarrow$) If Player II does not have a forced win in $G(u, v, k, n)$, then Player I does. Thus there is a move for Player I, say the pebble colored x_i to $a \in A$, such that for any move for Player II, say to $b \in B$, Player I has a forced win in the game $G(u[x_i/a], v[x_i/b], k, n-1)$. By the induction hypothesis, there is a formula $\psi_b \in L_{k,n-1}$ such that $A, u[x_i/a] \models \psi_b$ but $B, v[x_i/b] \models \neg\psi_b$.

By Lemmas 10 and 11, there are only finitely many inequivalent formulas of $L_{k,n-1}$. Thus the infinitary formula

$$\bigwedge_{b \in B} \psi_b$$

is expressible by a formula of $L_{k,n-1}$, and

$$\begin{aligned} A, u &\models \exists x_i \bigwedge_{b \in B} \psi_b \ , \\ B, v &\models \neg\exists x_i \bigwedge_{b \in B} \psi_b \ , \end{aligned}$$

and $\exists x_i \bigwedge_{b \in B} \psi_b$ is expressible by a formula of $L_{k,n}$. Therefore u and v are not $L_{k,n}$-equivalent. □

The following theorem is an immediate consequence of Lemmas 4 and 12.

Theorem 13 *Σ satisfies the k-variable property if and only if for all A, B models of Σ and k-configurations (u, v) over A, B, if Player II has a forced win in every game $G(u, v, k, n)$, $n \geq 0$, then Player II has a forced win in every game $G(u, v, m, n)$, $m \geq k$, $n \geq 0$.*

4 Three Variables are Necessary and Sufficient for Linear Order

In this section we give a single proof that encompasses the results of [11, 16], illustrating the Ehrenfeucht-Fraisse game of §3. We consider games played on linear orders with monadic predicates.

Theorem 14 *Linear order satisfies the 3-variable property and does not satisfy the 2-variable property.*

Proof. For the upper bound, by Theorem 13 it suffices to show that for any 3-configuration (u, v), if Player II has a forced win in $G(u, v, 3, n)$, then Player II has a forced win in $G(u, v, k, n)$, for all k. The result holds for any $k \leq 3$ by Lemma 9, so assume $k > 3$.

We will describe Player II's best strategy in $G(u, v, k, n)$ and prove the theorem by simultaneous induction on n. For $n = 0$, the assertion that Player II has a forced win in the game $G(u, v, k, 0)$ says that (u, v) is a local isomorphism, which follows immediately from the assumption that Player II has a forced win in the game $G(u, v, 3, 0)$.

Suppose now that $n > 0$. If $|u| = |v| < 3$, then for any move that Player I might make, let Player II respond according to an optimal strategy in the game $G(u, v, 3, n)$. If the resulting configuration is (u', v'), then by Definition 7, Player II has a forced win in the game $G(u', v', 3, n-1)$. By the induction hypothesis, Player II has a forced win in every game $G(u', v', k, n-1)$, $k > 0$. Again by Definition 7, since the move of Player I was arbitrary, Player II has a forced win in $G(u, v, k, n)$, $k > 0$.

If $|u| = |v| = 3$, renumber the variables if necessary so that $u(x_1) < u(x_2) < u(x_3)$ and $v(x_1) < v(x_2) < v(x_3)$. (Note (u, v) is a local isomorphism, since Player II has a forced win in $G(u, v, 3, 0)$. If some $u(x_i) = u(x_j)$, $i \neq j$, then a pair of pebbles can be removed, and we revert to the previous case.) Consider the pair of corresponding regions

$$\{a \in A \mid a < u(x_2)\}, \quad \{b \in B \mid b < v(x_2)\} .$$

Associate with this pair of regions the game

$$G(u_<, v_<, 3, n) ,$$

where $u_<$ and $v_<$ are u and v, respectively, restricted to domain $\{x_1, x_2\}$. Similarly, associate with the pair of corresponding regions

$$\{a \in A \mid a \geq u(x_2)\}, \quad \{b \in B \mid b \geq v(x_2)\}$$

the game

$$G(u_\geq, v_\geq, 3, n) ,$$

where $u_\geq$ and $v_\geq$ are u and v, respectively, restricted to domain $\{x_2, x_3\}$. By Lemma 9, Player II has forced wins in both of these games. But $|u_<| = |u_\geq| < 3$, so by a case previously considered, Player II has a forced win in the games $G(u_<, v_<, k, n)$ and $G(u_\geq, v_\geq, k, n)$, $k > 0$.

We now describe a strategy for Player II in the game $G(u, v, k, n)$. Assume $k > n$, so that Player I never needs to remove a pebble from the board. The result follows for smaller k by Lemma 9. Whenever Player I moves in one of the designated regions of either A or B, Player II responds with an optimal strategy in the game associated with that region. Player II will then move in the corresponding region in the other structure, since there is always a pebble on $u(x_2)$. If (u', v') is any subsequent (global) configuration, the restriction of (u', v') to either of the two pairs of regions is a local isomorphism, since Player II has a forced win in the game associated with that region. Moreover, all points of the region $\{a \in A \mid a < u(x_2)\}$ are less than all points of the region $\{a \in A \mid a \geq u(x_2)\}$, and similarly for $\{b \in B \mid b < v(x_2)\}$ and $\{b \in B \mid b \geq v(x_2)\}$. Therefore (u', v') is a local isomorphism. This establishes the upper bound.

To show that two variables do not suffice, we observe that $\mathcal{Z}$ and $\mathcal{Q}$, without monadic predicates, are L_2-equivalent but not L_3-equivalent. This follows from Theorem 13 and Example 8. □

5 Theories of Bounded-Degree Trees

In this section we define various theories of bounded-degree trees and establish tight upper and lower bounds on the number of bound variables needed to define any first-order definable formula.

Definition 15 Consider a language L with a binary relation symbol $\leq$ and equality $=$, possibly with extra monadic predicates. Let L^+ be L augmented with a binary function symbol $+$. The atomic formula $x \leq y$ is read, "x is a descendant of y" or "y is an ancestor of x." The function $+$ is intended to give the least common ancestor (LCA), or least upper bound with respect to $\leq$.

Consider the following axioms of L and L^+. The axioms (i)-(vd) are expressed in the language L, and (vi) is expressed in the language L^+.

(i) "$\leq$ is a partial order."

(ii) "$\leq$ is a linear order above any x."

(iii) "Every pair x, y has an LCA."

(iv$d$) "There is no set of $d+1$ proper descendants of x whose pairwise LCA is x."

(vd) "Every non-leaf x has a set of d proper descendants whose pairwise LCA is x."

(vi) "$x + y$ is the LCA of x and y."

The theories $S_d \subseteq L$ and $S_d^+ \subseteq L^+$ describe trees of degree d. S_d^+ consists of axioms (i)-(vi), and S_d is obtained from S_d^+ by omitting (vi). The theories $T_d \subseteq L$ and $T_d^+ \subseteq L^+$ describe trees of degree at most d. These theories are obtained from S_d and S_d^+, respectively, by omitting axiom (vd). □

Note that models of these theories need not be discrete; there is no notion of "child" or "parent".

For A a model of T_d, $a, a_1, a_2 \in A$, $a_1, a_2 < a$, define $a_1 \equiv_a a_2$ if $a_1 + a_2 < a$. It follows from the axioms of T_d that $\equiv_a$ is an equivalence relation with at most d equivalence classes, and exactly d if A is a model of S_d and a is not a leaf. These classes are called *subtrees* of a. For $a' < a$, denote the subtree of a containing a' by $T(a, a')$. Denote the $L_{k,n}$-type of the valuation $x_1 \mapsto a$, $x_2 \mapsto a'$ by $\theta_{k,n}(a, a')$. This is a set of formulas of $L_{k,n}$ with free variables among x_1, x_2. For a subtree T of a, define

$$\Theta_{k,n}(T) \quad = \quad \{\theta_{k,n}(a, a') \mid a' \in T\} \ .$$

Lemma 16 *Let Σ be one of the theories T_d, T_d^+, S_d, S_d^+. Let (u,v) be any k-configuration, and let a and b be the suprema of $\{u(x) \mid x \in \partial u\}$ and $\{v(x) \mid x \in \partial v\}$, respectively. Let $f(n)$ be any sufficiently fast-growing function of n, and let*

$$k \;=\; \begin{cases} \max(4,d)\ , & \text{if } \Sigma = T_d \text{ or } T_d^+ , \\ \max(4, \lceil \frac{d}{2} \rceil)\ , & \text{if } \Sigma = S_d \text{ or } S_d^+ \ . \end{cases}$$

If Player II has a forced win in $G(u,v,k,f(n))$, then there is a one-one correspondence between subtrees $A_1,\ldots,A_l$ of a and subtrees $B_1,\ldots,B_l$ of b such that

(i) $\Theta_{k,n}(A_i) = \Theta_{k,n}(B_i)$, $1 \le i \le l$,

(ii) $u(x) \in A_j$ *iff* $v(x) \in B_j$.

Proof. For the case of T_d or T_d^+, we show first that a and b have the same number of subtrees. If not, suppose a has more than b. We will describe a winning strategy for Player I, contradicting the assumption that Player II has a forced win in $G(u,v,k,f(n))$. Let Player I play a pebble on a, if a is not already covered. Then Player II must play b, otherwise Player I wins in at most one move. (It follows from T_d that $a = u(x) + u(y)$ for some $x, y \in \partial u$, and $b = v(x) + v(y)$.) Now let Player I successively play pebbles in as many distinct subtrees of a as possible, leaving a pebble on a. Player II must respond by pebbling in separate subtrees of b, otherwise Player I wins in at most one move. Thus, if the number of subtrees of a is less than d, or if $d < 4$, we are done. Otherwise, since there are d pebbles, there must be d subtrees of a, of which $d-1$ have pebbles, and $d-1$ subtrees of b, all of which have pebbles. Now Player I removes the pebble from a and places it somewhere in the last subtree of a. Player II must play the corresponding pebble on a point in one of the subtrees of b, otherwise Player I wins immediately. Now there exist pebbled points b_0, b_1, b_2 such that $b_0 + b_1 < b_1 + b_2$, whereas for the corresponding points a_0, a_1, a_2, $a_0 + a_1 = a_1 + a_2$. Thus Player I wins in one more move by pebbling $b_0 + b_1$ with a fourth pebble, keeping b_0, b_1, and b_2 pebbled. Note that $k = \max(4,d)$ pebbles are required for this argument.

The remainder of the argument uses only $\max(4, \lceil \frac{d}{2} \rceil)$ pebbles and works for all four theories under consideration. By the preceding paragraph, we may assume that a and b have the same number of subtrees. We show (i) first. Suppose there is a Θ such that the number of subtrees T of a with $\Theta_{k,n}(T) = \Theta$ is different from the number of subtrees of b satisfying this property. We will again describe a winning strategy for Player I. Note that there must be some $r < \lceil \frac{d}{2} \rceil$ and Θ such that exactly r subtrees T of b have $\Theta_{k,n}(T) = \Theta$ and the number of subtrees of a satisfying this property is strictly greater than r, or vice-versa (without loss of generality, assume the former). As above, Player I will pebble in $r+1$ subtrees of a of type Θ, and Player II will be forced to play in a subtree of b of type different from Θ. Now there are pebbles on a', b' and at least two other points a'', b'' such that $a' + a'' = a$ and $b' + b'' = b$, and

$$\Theta_{k,n}(T(a,a')) \;\neq\; \Theta_{k,n}(T(b,b'))\ .$$

Let $a'' \in T(a,a')$ such that for no $b'' \in T(b,b')$ is $\theta_{k,n}(a,a'') = \theta_{k,n}(b,b'')$. Player I plays a if not already played. Player II must respond with b, otherwise Player I wins in 1 move. Player I now plays a''. Player II must respond with some $b'' < b$, but whatever b'' is played,

$$\theta_{k,n}(a,a'') \quad \neq \quad \theta_{k,n}(b,b'') \ .$$

By Lemma 12, Player I has a forced win in $G((a,a''),(b,b''),k,n)$ and therefore also in $G(u,v,k,f(n))$, by Lemma 9. This is a contradiction.

Finally, we show that

$$\Theta_{k,n}(T(a,u(x))) \quad = \quad \Theta_{k,n}(T(b,v(x))) \ .$$

If not, let $a' \in T(a,u(x_i))$ such that $\theta_{k,n}(a,a') \notin \Theta_{k,n}(T(b,v(x_i)))$. (The opposite case is symmetric.) Player I pebbles a' with a pebble of color other than x_i, and the argument now proceeds as in the preceding paragraph. □

Definition 17 For a theory Σ, define $var(\Sigma)$ to be the minimum k such that Σ has the k-variable property, if such a k exists, or ∞ otherwise. □

The following results determine $var(\Sigma)$ exactly for T_d, T_d^+, S_d, and S_d^+.

Theorem 18

$$var(T_d) = var(T_d^+) = \begin{cases} 3 \ , & \textit{if } d = 1, \\ \max(4,d) \ , & \textit{if } d > 1 \ . \end{cases}$$

$$var(S_d) = var(S_d^+) = \begin{cases} 3 \ , & \textit{if } d = 1, \\ \max(4,\lceil \frac{d}{2} \rceil) \ , & \textit{if } d > 1 \ . \end{cases}$$

Proof. The bounds for $d = 1$ were proved in Theorem 14. For $d > 1$ and for Σ any of the four theories we are considering, define

$$k \quad = \quad \begin{cases} \max(4,d) \ , & \text{if } \Sigma = T_d \text{ or } T_d^+, \\ \max(4,\lceil \frac{d}{2} \rceil) \ , & \text{if } \Sigma = S_d \text{ or } S_d^+ \ . \end{cases}$$

Let g and h be sufficiently fast-growing functions of n such that $g(n) \gg f(h(n))$ and $h(n) \gg g(n-1)$, where f is the function of Lemma 16.

We must show that for all A and B satisfying Σ, and for all k-configurations (u,v) over A and B, if Player II has a forced win in the games $G(u,v,k,n)$ for all n, then Player II has a forced win in the games $G(u,v,m,n)$ for all m and n. We actually show by induction on n that if Player II has a forced win in the game $G(u,v,k,g(n))$, then Player II has a forced win in the games $G(u,v,m,n)$ for all m.

As in Theorem 14, the basis $n = 0$ is immediate. Suppose now that the theorem holds for $n-1$. Assume $m > n$, so that Player I will never have to remove a pebble from the board. The result will follow for smaller values of m by Lemma 9. If $|u| = |v| < k$, let Player II respond to any move of Player I with an optimal move according to Player II's winning strategy in $G(u, v, k, g(n))$. By Definition 7, if the resulting configuration is (u', v'), then Player II has a forced win in $G(u', v', k, g(n)-1)$, and $g(n)-1 > g(n-1)$. By Lemma 9 and the induction hypothesis, Player II has a forced win in $G(u', v', m, n-1)$ for all m. Since Player I's move was arbitrary, this constitutes a forced win for Player II in $G(u, v, m, n)$.

Now suppose $|u| = |v| = k$. As in the proof of Theorem 14, we will break the game $G(u, v, m, n)$ up into several smaller games on which Player II has a forced win, and combine these strategies to produce a winning strategy for Player II on $G(u, v, m, n)$.

Let $\langle u \rangle$ be the smallest subset of A containing all the $u(x)$ and closed under the operation $+$. Let $\langle v \rangle$ be the corresponding set in B. Let $a \in \langle u \rangle$, and let b be the corresponding element of $\langle v \rangle$. Let $A_1, \ldots, A_l$ be the subtrees of a, and let $B_1, \ldots, B_l$ be the subtrees of b, such that

$$\Theta_{k,h(n)}(A_i) \quad = \quad \Theta_{k,h(n)}(B_i) \ , \quad 1 \le i \le l \ ,$$

and such that $u(x) \in A_i$ iff $v(x) \in B_i$. Such a correspondence between the subtrees of a and b exists, by Lemma 16.

If $\langle u \rangle \cap A_i \neq \emptyset$, let $a_i < a$ be its supremum. Let $b_i < b$ be the corresponding element of $\langle v \rangle \cap B_i$. In this case we associate the game

$$G((a, a_i), (b, b_i), k, h(n)) \tag{4}$$

with the pair of regions

$$A_i - \{a' \mid a' < a_i\} \ , \quad B_i - \{b' \mid b' < b_i\} \ .$$

If $\langle u \rangle \cap A_i = \emptyset$, let $a_i \in A_i$ and $b_i \in B_i$ such that

$$\theta_{k,h(n)}(a, a_i) \quad = \quad \theta_{k,h(n)}(b, b_i) \ .$$

In this case we associate the game

$$G((a, a_i), (b, b_i), k, h(n)) \tag{5}$$

with the pair of corresponding regions A_i and B_i. Finally, if a and b are the suprema of $\langle u \rangle$ and $\langle v \rangle$, respectively, then we associate the game

$$G((a), (b), k, h(n)) \tag{6}$$

with the regions

$$A - \{a' \mid a' < a\} \ , \quad B - \{b' \mid b' < b\} \ .$$

Observe that the designated regions of A partition A, and those of B partition B. We claim that Player II has a forced win in each of the games

$$G(u', v', k, h(n))$$

associated with these regions. In case 5, this is because of the property 5 and Lemma 12; in cases 4 and 6, it is because with k pebbles and at most two moves, Player I can force the configuration (u', v') from (u, v) as long as Player II is playing optimally, thus Player II has a forced win in $G(u', v', k, g(n) - 2)$ and $g(n) - 2 > h(n)$. (Here we are using the property that any uncovered $a \in \langle u \rangle$ is $u(x) + u(y)$ for some $x, y \in \partial u$.) Moreover, $|u'| = |v'| < k$. Therefore, by a previous case, Player II has a forced win in the games

$$G(u', v', m, n)$$

for all m. We now combine optimal strategies for Player II in all these games, as in Theorem 14. Whenever Player I plays in one of the designated regions, Player II responds in the corresponding region of the other structure, according to his best strategy in the game associated with that region. Player II's play will always be in the correct region, since pebbles are never removed. Let (u_t, v_t) be the sequence of configurations. Since Player II has a forced win in each of these games, each (u_t, v_t) restricted to each region is a local isomorphism; and by the choice of regions, if $u_t(x)$ and $u_t(y)$ are in different regions, then $u_t(x) \leq u_t(y)$ iff $v_t(x) \leq v_t(y)$.

The lower bounds follow from the lower bounds for linear order (Theorem 14), the lower bounds for finite trees (Theorem 19), and the following argument that all of $var(T_d)$, $var(T_d^+)$, $var(S_d)$, and $var(S_d^+)$ are at least 4 for $d > 1$.

Let $d > 1$ and let A be a full d-ary tree such that each path has order type $\mathcal{Z} + \mathcal{Z}$. (Here "+" denotes the partial order obtained by placing the two operands end-to-end.) Let a_3 and a_4 be two vertices in the lower part of A whose LCA is a_2 in the upper part of A. Thus the order type of the path from a_2 to a_3 and from a_2 to a_4 is $\omega + \omega^R$. Let a_1 be the parent of a_2, and let a_0 be the parent of a_1.

Let $B = A$ and consider the game

$$G((a_0, a_3, a_4), (a_1, a_3, a_4), 3, n) \ .$$

Player II has a forced win, since the initial configuration (u, v) is a local isomorphism (even in the presence of $+$), and for subsequent moves, as soon as Player I picks up a pebble, there is an automorphism of A sending the remaining two points on the left to the remaining two points on the right. From then on, Player II can always play the image under that automorphism or its inverse of the point that Player I plays.

However, Player I has a forced win in

$$G((a_0, a_3, a_4), (a_1, a_3, a_4), 4, 2) \ .$$

Player I first pebbles $a_2 = a_3 + a_4$ with the spare pebble on the left, to which Player II must respond with a_2 with the spare pebble on the right; then Player I removes the pebble on a_3 on the left and plays it on a_1, to which Player II has no response. $\square$

6 Finite models

It is interesting to note how the situation changes when we restrict our attention to finite models. Not only does $var(\cdot)$ change, but the models are definable up to isomorphism. In this case we can give direct proofs of the upper bounds without using the Ehrenfeucht-Fraisse games of §3.

Define $Fin(\Sigma)$ to be the set of finite models of the theory Σ. For a set of structures S, define $var(S)$ to be the minimum k such that S satisfies Corollary 4(i).

Theorem 19 *(i)* $var(Fin(T_d)) = \begin{cases} 2\ , & \textit{if } d = 1, \\ \max(3, d)\ , & \textit{if } d > 1\ . \end{cases}$

(ii) $var(Fin(S_d)) = \begin{cases} 2\ , & \textit{if } d = 1 \textit{ or } 2, \\ \max(3, \lceil \frac{d}{2} \rceil\ , & \textit{if } d > 2\ . \end{cases}$

Proof. We first establish the upper bounds. Given a finite tree A, we produce a formula $\varphi_A(x)$ such that, whenever B is a tree and $b \in B$, then $\varphi_A(b)$ holds in B iff the subtree of B with root b is isomorphic to A.

If A consists of a single node a, we assert that x satisfies the same monadic predicates that a does, and that x has no proper descendants. This takes two variables.

If A contains at least two nodes, let a be the root of A and let $A_1, \ldots, A_m$ be the maximal proper subtrees of A, with roots $a_1, \ldots, a_m$, respectively. Assume by induction on height that the formulas $\varphi_{A_1}(x), \ldots, \varphi_{A_m}(x)$ have been constructed. As above, we first assert that x satisfies the same monadic predicates that a does. We then assert that there exist proper subtrees satisfying each of the $\varphi_{A_i}(x)$:

$$\bigwedge_{i=1}^{m} \exists y < x\ \varphi_{A_i}(y)$$

and that every proper subtree has a supertree satisfying one of them:

$$\forall y < x\ \exists x \geq y\ \bigvee_{i=1}^{m}\ \varphi_{A_i}(x)\ .$$

Each of these statements takes two variables. Together they establish the isomorphism types of the maximal proper subtrees. For the cases T_1 and S_1, we are done; for the other cases, it remains to show how to specify the number of maximal proper subtrees of each isomorphism type.

Consider the case T_d. Observe that the predicate "y is a child of z" is expressible with three variables:

$$y < z \wedge \forall x\ (y < x \rightarrow z \leq x)\ .$$

So is the predicate "y and z are siblings":

$$\forall x\ (y \text{ is a child of } x \leftrightarrow z \text{ is a child of } x)\ .$$

If there are $p < d$ children of a of isomorphism type A_i, we can express this with the formula

$$\begin{array}{ll} \exists y_1 \ldots \exists y_p & \text{the } y_j \text{ are distinct children of } x \\ \wedge & \bigwedge_{j=1}^{p}\ \varphi_{A_i}(y_j) \\ \wedge & \forall x\ (x \text{ and } y_1 \text{ are siblings } \wedge \varphi_{A_i}(x)) \rightarrow \bigvee_{j=1}^{p}\ y_j = x\ . \end{array}$$

This takes $\max(3, p+1) \leq d$ variables. The only case of T_d not yet covered is when a has d maximal proper subtrees, all isomorphic to A_1. In this case we say that all children y of x satisfy $\varphi_{A_1}(y)$, and there is a child of x with $d-1$ distinct siblings. The variable x can be used to name one of the siblings. This takes $\max(3, d)$ variables. This completes the argument for T_d.

Now consider the case S_d. Above, we specified all the isomorphism types of the maximal proper subtrees A_i of A. If all the A_i are isomorphic, or if all are of distinct isomorphism types, we are done. This exhausts the case $d = 2$, so let $d \geq 3$. We will show that $\max(3, \lceil \frac{d}{2} \rceil)$ variables are sufficient. If all isomorphism types are satisfied by at most $\lceil \frac{d}{2} \rceil$ of the A_i, then as above we can use $\lceil \frac{d}{2} \rceil$ variables y_j to name those a_i and assert $\varphi_{A_i}(y_j)$. This takes $\max(3, \lceil \frac{d}{2} \rceil)$ variables. If there are $m > \lceil \frac{d}{2} \rceil$ A_i isomorphic to A_1, then all other isomorphism types are covered by the preceding case; we can then assert that there are $d-m < \lfloor \frac{d}{2} \rfloor$ distinct children of x *not* satisfying φ_{A_1}, and then use one universally quantified variable to assert that all other siblings satisfy φ_{A_1}. In either case we have used $\max(3, \lceil \frac{d}{2} \rceil)$ variables.

We now establish the corresponding lower bounds. In each of the cases below, we produce two finite models A and B of one of our theories Σ such that A and B are not elementarily equivalent, but A and B are L_v equivalent, where $v = var(Fin(\Sigma)) - 1$. In each of the cases below, it is fairly straightforward to determine a winning strategy for Player II for the game $G(\emptyset, \emptyset, v, n)$. We leave this to the reader.

(Case T_1, S_1, S_2, $v = 1$.) Let A and B be arbitrary nonempty nonisomorphic models with no monadic predicates.

(Case T_2, $v = 2$.) Let A be the complete binary tree of depth 2, and let B be the binary tree of depth 2 with two nodes of depth 1, three leaves of depth 2, and no monadic predicates.

(Case T_d, $v = d-1$.) Let A be the complete d-ary tree of depth 1, B the complete $(d-1)$-ary tree of depth 1, and no monadic predicates.

(Case S_3, S_4, $v = 2$.) Let A and B be complete d-ary trees of depth 2. Let M be a monadic predicate such that one of each cluster of leaves satisfies M in A, and two of each cluster satisfy M in B.

(Case S_{2k}, $k > 2$, $v = k-1$.) Let B and C be complete $2k$-ary trees of depth 1. Let M be a monadic predicate true of exactly $k-1$ leaves in A and k leaves in B.

(Case S_{2k+1}, $k > 1$, $v = k$.) Let A and B be complete $(2k+1)$-ary trees of depth 1. Let M be a monadic predicate true of exactly k leaves in A and $k+1$ leaves in B. □

7 Conclusion

Some interesting questions remain. One is to establish a general model-theoretic characterization of those relational structures that possess the k-variable property for some k. Another is to give *natural* complete sets of temporal connectives for branching-time models of temporal logic, whose existence is implied by Gabbay's result [7] and the results of this paper.

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