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Research Report

ON THE REPRESENTATION OF DYNAMIC ALGEBRAS

Dexter Kozen

IBM Thomas J. Watson Research Center
Yorktown Heights, New York 10598

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ABSTRACT

It has been shown that all separable dynamic algebras are represented by (possibly nonstandard) Kripke models of propositional dynamic logic. In this paper a separable dynamic algebra is constructed which is not represented by any standard Kripke model. The proof makes essential use of the duality of dynamic algebras and topological Kripke models.

Introduction

In [K1] we gave a representation theorem for dynamic algebras: any separable dynamic algebra is isomorpnic to the characteristic algebra of a (possibly nonstandard) Kripke model of propositional dynamic logic (PDL). A *standard* Kripke model is one for which α^* must be α^{rtc}, the reflexive transitive closure of α. In this note we construct a separable dynamic algebra not isomorphic to the characteristic algebra of any standard Kripke model. The construction of the counterexample uses the notion of *topological Kripke model* introduced in [K2], where a duality between certain topological Kripke models and separable dynamic algebras was exhibited. This duality is fully exploited here.

The * operator of PDL is a traditional source of difficulty in completeness proofs, since PDL is not compact when interpreted over standard Kripke models. For example, the set

$$\{ <\alpha^*>X \} \cup \{ \neg X, \neg<\alpha>X, \neg<\alpha^2>X, \ldots \}$$

has no model, yet every finite subset does. However, the logic becomes compact when nonstandard Kripke models are introduced, and completeness proofs become easier, as demonstrated by Berman [Be]. This paper shows that the inclusion of nonstandard models is not entirely without loss of generality.

An understanding of the * operator requires an understanding of the discrepancy between the standard and nonstandard Kripke models. In [K2] several results were proved which lent insight into the nature of the * operator and this discrepancy. In particular, it was shown that any set $<\alpha^*>X - <\alpha^{rtc}>X$ is nowhere dense. In a standard Kripke model all such sets are empty, and in fact any Kripke model may be made standard by removing these points. However, the resulting space is no longer compact and need not have the same characteristic algebra as the original. In fact, the counterexample constructed here involves the construction of a nonstandard Kripke model in which *every* point is in some $<\alpha^*>X - <\alpha^{rtc}>X$. It is built on the Cantor space, a traditional source of counterexamples in topology.

It is assumed that the reader is familiar with dynamic logic and at least casually familiar with the contents of [K1,K2]. A review of the definitions and some of the results is included here for completeness, but [K1,K2] should be consulted for a more thorough treatment.

Dynamic algebras

A *Kleene algebra* (or *relation algebra*) K is a structure

$$K = (K, \cup, 0, ;, \lambda, ^-, ^*)$$

such that $(K, \cup, 0)$ is an upper semilattice with identity 0, $(K, ;, \lambda)$ is a monoid, and $^-$ and * are unary operations satisfying the axioms

$$\alpha;(\beta \cup \gamma) = \alpha;\beta \cup \alpha;\gamma,$$
$$(\alpha \cup \beta);\gamma = \alpha;\gamma \cup \beta;\gamma,$$
$$\alpha;0 = 0;\alpha = 0,$$
$$(\alpha;\beta)^- = \beta^-;\alpha^-,$$
$$(\alpha \cup \beta)^- = \alpha^- \cup \beta^-,$$
$$\alpha^{--} = \alpha,$$

(1) $$\alpha;\beta^*;\gamma = \sup_n \alpha;\beta^n;\gamma,$$

where in (1), $\alpha^0 = \lambda$, $\alpha^{n+1} = \alpha;\alpha^n$, and the supremum is with respect to the semilattice order $\leq$ in K.

A *dynamic algebra* is a structure

$$D = (K, B, <>)$$

where K is a Kleene algebra, B is a Boolean algebra, and $<>$ is a scalar multiplication satisfying the following axioms:

$$<\alpha \cup \beta>X = <\alpha>X \vee <\beta>X$$
$$<\alpha>(X \vee Y) = <\alpha>X \vee <\alpha>Y$$
$$<\alpha>(<\beta>X) = <\alpha\beta>X$$
$$<\alpha>0 = <0>X = 0$$
$$<\lambda>X = X$$

(2) $$X \leq [\alpha]<\alpha^->X$$

(3) $$<\alpha^*>X = \sup_n <\alpha^n>X$$

where in (2), $[\alpha]X$ denotes the dual multiplication

$$[\alpha]X = \neg<\alpha>\neg X$$

and in (3), the supremum is with respect to the lattice order $\leq$ in B.

Kripke models

A *Kripke model* is a structure

$\mathcal{A} = (S,K,B)$

where S is a nonempty set of *states*, B is a Boolean algebra of subsets of S with the set-theoretic Boolean algebra operations, and K is a Kleene algebra of relations on S in which ∪ is set union in K, ; is relational composition, λ is the identity relation, 0 is the null set, ⁻ gives the reverse of a relation, and * is any unary relation satisfying (1) and (3), such that the set

(4) $$<\alpha>X = \{ s \mid \exists t \in X\ (s,t) \in \alpha \}$$

is in B whenever $\alpha \in K$ and $X \in B$. If (4) is taken as the definition of scalar multiplication <>, then the structure (K,B,<>) is a dynamic algebra, called the *characteristic algebra* of $\mathcal{A}$ and denoted $C(\mathcal{A})$. A Kripke model is called *standard* if $\alpha^* = \alpha^{rtc}$ for any $\alpha \in K$, where

$$\alpha^{rtc} = \bigcup_n \alpha^n$$

is the reflexive transitive closure of α; otherwise it is called *nonstandard*. In any Kripke model, α^* is a reflexive, transitive relation containing α, and is the least element of K satisfying this property. Thus in a nonstandard model, α^* contains α^{rtc}, but they need not be equal.

Separability

A dynamic algebra D = (K,B,<>) is *separable* if for every $\alpha, \beta \in K$, $\alpha \neq \beta$, there exists an $X \in B$ such that $<\alpha>X \neq <\beta>X$. A Kleene algebra K is *separable* if there exists a separable dynamic algebra over K. Not all Kleene algebras are separable [K1]. Pratt [Pr3 ...] discussed the importance of this property.

Representation of separable dynamic algebras as nonstandard Kripke models

In [K1] it was shown that if K is separable, then any dynamic algebra D = (K,B,<>) is isomorphic to $C(\mathcal{A})$ for some (possibly nonstandard) Kripke model $\mathcal{A}$. We outline here the slightly weaker result that any separable dynamic algebra is isomorphic to $C(\mathcal{A})$ for some (possibly nonstandard) $\mathcal{A}$. The construction uses a technique similar to the Stone representation theorem for Boolean algebras. Let U, V denote ultrafilters of B, and let S be the set of all ultrafilters. Let

$$X' = \{ U \mid X \in U \},\ B' = \{ X' \mid X \in B \},$$
$$\alpha' = \{ (U,V) \mid \forall X \in V\ <\alpha>X \in U \},\ K' = \{ \alpha' \mid \alpha \in K \},$$

and let $S(D)$ be the structure

$\mathcal{S}(D) = (S, K', B')$.

As shown in [K1], $\mathcal{S}(D)$ is a Kripke model and the map $'$ is a dynamic algebra isomorphism $D \to \mathcal{C}(\mathcal{S}(D))$.

Topological Kripke models

A *topological Kripke model* is a Kripke model $\mathcal{A} = (S, K, B)$ with topology on S generated by B.

If D is a separable dynamic algebra, the Kripke model $\mathcal{S}(D)$ with this topology is called the *Stone space* of D, by analogy with the Stone duality for Boolean algebras.

A topological Kripke model has a base of closed and open (clopen) sets, namely the elements of B. If $\mathcal{A}$ is compact, then the clopen sets are exactly the elements of B [BS, Lemma 1.6.4]. The topology on S induces a product topology on S × S; terms such as open, closed, etc. applied to subsets of S × S refer to this topology. cl(X) and cl(α) denote topological closure.

In the following, $\mathcal{A} = (S, K, B)$ is a topological Kripke model, α, β denote subsets of S × S, and X, Y denote subsets of S. If P is any property of sets, we say that $\langle\alpha\rangle$ *preserves P sets* if $\langle\alpha\rangle$X is *P* whenever X is *P*. The definition for $[\alpha]$ is similar.

Note that $\langle\alpha\rangle$ preserves open (closed) sets iff $[\alpha]$ preserves closed (open) sets, and if $\alpha \in K$ and $\mathcal{A}$ is compact, then $\langle\alpha\rangle$ and $[\alpha]$ both preserve clopen sets.

The following elementary properties were proved in [K2].

Proposition 5. The following are equivalent:

(i) $\langle\alpha\rangle$ preserves open sets,

(ii) $\langle\alpha\rangle$X is open whenever X is clopen.

Proposition 6. Suppose $\langle\alpha\rangle$ preserves closed sets and $\langle\alpha^-\rangle$ preserves open sets. Then

$$\langle\alpha\rangle\mathrm{cl}(X) = \mathrm{cl}(\langle\alpha\rangle X).$$

Proposition 7. Suppose A is compact and both $<\alpha>$, $<\alpha^->$ preserve closed sets. If F is a directed family of closed subsets of S (that is, whenever X, Y $\in$ F, there exists a Z $\in$ F such that Z $\subseteq$ X $\cap$ Y), then

$$\cap_F <\alpha>X = cl(<\alpha> \cap_F X).$$

Proposition 8. Suppose A is compact and Hausdorff. Then the following are equivalent:

(i) $<\alpha>$, $<\alpha^->$ preserve closed sets,

(ii) α is closed.

Proposition 9. If A is compact and Hausdorff and X is a clopen set, then for any α, $<cl(\alpha)>X = cl(<\alpha>X)$.

Proposition 10. Let A be compact and Hausdorff. If α, β are closed, and if $<\alpha>X = <\beta>X$ for all clopen X, then $\alpha = \beta$.

Dynamic spaces and their duality with separable dynamic algebras

A *dynamic space* is a topological Kripke model A = (S,K,B) such that

(i) A is compact and Hausdorff

(ii) all elements of K are closed.

The following three results, proved in [K2], capture the duality between separable dynamic algebras and dynamic spaces.

Theorem 11. If D is a separable dynamic algebra then S(D) is a dynamic space.

Theorem 12. If A is a dynamic space then $C(A)$ is a separable dynamic algebra.

Theorem 13. (i) If D is a separable dynamic algebra, then D is isomorphic to $C(S(D))$.
(ii) If A is a dynamic space, then A is homeomorphic to $S(C(A))$.

Further properties of dynamic spaces

Let A = (S,K,B) be a dynamic space.

Proposition 14. For all $\alpha \in$ K, $\alpha^* = cl(\alpha^{rtc})$.

Proposition 15. Let $\alpha \in$ K, X $\in$ B. The following statements are equivalent:

(i) $<\alpha^{rtc}>X = <\alpha^*>X$,

(ii) $<\alpha^{rtc}>X$ is closed,

(iii) $<\alpha^{rtc}>X = \bigcup_{n \leq m} <\alpha^n>X$ for some $m < \omega$.

Proposition 16. For any $\alpha \in K$, $X \in B$, the set $<\alpha^*>X - <\alpha^{rtc}>X$ is nowhere dense.

A counterexample

In this section we construct a separable dynamic algebra not isomorphic to $C(A)$ for any standard Kripke model A. The algebra we construct will actually be the characteristic algebra of a nonstandard topological Kripke model.

Let $\Sigma = \{0,1\}$ have the discrete topology and let Σ^ω, the set of countable sequences over Σ, have the product topology. Σ^ω with this topology is often called the *Cantor space*. The topology is most easily seen in terms of a metric: for s, t $\in \Sigma^\omega$, let

$$d(s,t) = 2^{-|gcs(s,t)|},$$

where gcs(s,t) denotes the greatest common initial substring of s and t, and $|w|$ denotes the length of a string w. Then the metric topology and the Cantor space topology coincide. It is easily shown that Σ^ω is compact, Hausdorff, totally disconnected, and has a countable basis of clopen neighborhoods

$$X_w = \{ s \mid w \text{ is an initial substring of } s \},$$

one for each word w of finite length over Σ.

For each $s \in \Sigma^\omega$, define

$$\alpha_s = \{ (u,v) \mid d(s,v) \leq 2d(s,u) \}.$$

That is, u can go to v under α_s iff the distance from s is at most doubled.

Lemma 17. $<\alpha_s>$ and $<\alpha_s^->$ preserve both open and closed sets.

Proof. To show $<\alpha_s>$ and $<\alpha_s^->$ preserve closed sets, by Proposition 8 it suffices to show α_s is closed. If $(u_i,v_i) \rightarrow (u,v)$ and $(u_i,v_i) \in \alpha_s$, then $d(s,v_i) \leq 2d(s,u_i)$ and so $d(s,v) \leq 2d(s,u)$ by the continuity of d, therefore α_s is closed. To show $<\alpha_s>$ and $<\alpha_s^->$ preserve open sets, by Proposition 5 it suffices to show $<\alpha_s>X_w$ and $<\alpha_s^->X_w$ are open for every basic clopen set X_w. This follows immediately from the definition of α_s. □

We now construct a nonstandard Kripke model $\mathcal{A}$. The states of $\mathcal{A}$ will be Σ^ω. The Boolean algebra B will be the family of basic clopen sets X_w. The Kleene algebra K will be the algebra generated by all the α_s under the operations λ, 0, $\cup$, ;, $^-$, and *, where λ is the identity relation, 0 is the null set, $\cup$ is set union, ; is relational composition, $^-$ is the reverse operator, and α^* is defined to be $\Sigma^\omega \times \Sigma^\omega$ for all α not 0 or λ.

Lemma 18. K is a Kleene algebra, and all elements of K are closed.

First we show that all elements of K are closed. The generators α_s are closed by Lemma 17, λ and 0 are closed, and α^* is closed for any α, so it suffices to show that ;, $^-$, and $\cup$ preserve closed sets. $\cup$ and $^-$ certainly do; to see that ; does, note that if α, β preserve closed sets, then so does $\alpha;\beta$. Then $\alpha;\beta$ is closed by Proposition 8.

We must now show that K is a Kleene algebra. K is surely a *-free Kleene algebra, since it is an algebra of binary relations under $\cup$, $^-$, and ;. Thus we have only to show that the operator * behaves properly, that is,

$$\alpha;\beta^*;\gamma = \sup_n \alpha;\beta^n;\gamma.$$

If $\alpha = 0$ or $\gamma = 0$, or if $\beta = 0$ or λ, then this condition holds trivially. Otherwise, it can be shown inductively that $\lambda \leq \alpha$ and $\lambda \leq \gamma$. This is because $\lambda \leq \alpha_s$ for any of the generators α_s of K by definition, and composition, reverse, or union of nonzero elements preserve this property. Similarly, it can be shown inductively that there exists a generator α_s with either $\alpha_s \leq \beta$ or $\alpha_s^- \leq \beta$; say $\alpha_s \leq \beta$ without loss of generality. Now we claim that $\alpha_s^* = \sup_n \alpha_s^n$. This claim establishes the result, since for any n,

$$\lambda;\alpha_s^n;\lambda \leq \alpha;\beta^n;\gamma,$$

therefore

$$\begin{aligned} \Sigma^\omega \times \Sigma^\omega &= \alpha_s^* \\ &= \sup_n \lambda;\alpha_s^n;\lambda \\ &= \sup_n \alpha;\beta^n;\gamma \\ &= \alpha;\beta^*;\gamma. \end{aligned}$$

In order to show $\alpha_s^* = \sup_n \alpha_s^n$, note that

$$\alpha_s^{rtc} = \{ (u,v) \mid u = s \rightarrow v = s \}$$

by definition of α_s, and therefore

$$\mathrm{cl}(\alpha_s^{\mathrm{rtc}}) = \Sigma^\omega \times \Sigma^\omega = \alpha_s^*.$$

Since all elements of K are closed, α_s^* is the smallest element of K containing α_s^{rtc}; in other words, α_s^* is the supremum of the α^n. □

Lemma 19. A is a dynamic space.

Proof. A is compact and Hausdorff and by the previous lemma, all elements of K are closed. Therefore it suffices to show that $C(A)$ is a dynamic algebra. By the previous lemma and Proposition 8, all elements of K preserve closed sets. Using Lemma 17 as basis, it is easy to show inductively that all elements of K preserve open sets. Thus all elements of K preserve clopen sets, i.e. $<\alpha>X \in B$ whenever $\alpha \in K$ and $X \in B$. Thus $(K,B,<>)$ is a $*$-free dynamic algebra. It remains to show that $*$ behaves properly, i.e.

$$<\alpha^*>X = \sup_n <\alpha^n>X.$$

But by Proposition 9

$$<\alpha^*>X = <\mathrm{cl}(\alpha^{\mathrm{rtc}})>X = \mathrm{cl}(<\alpha^{\mathrm{rtc}}>X),$$

and the result follows. □

Since A is a dynamic space, by Theorem 12 its characteristic algebra $C(A)$ is a separable dynamic algebra. We now show that this algebra may not be represented as the characteristic algebra of any standard Kripke model.

Theorem 20. $C(A)$ is not isomorphic to the characteristic algebra of any standard Kripke model.

Proof. Let $C(A) = (K,B,<>)$. By Theorem 13, the points of A are in one to one correspondence with the ultrafilters of B under the map Th defined by

$$\mathrm{Th}(s) = \{ X \in B \mid s \in X \}.$$

Call an ultrafilter U of B *$*$-consistent* if

(2.) $\quad <\alpha^*>X \in U$ iff $\exists n <\alpha^n>X \in U$

for any $\alpha \in K$, $X \in B$. Then no ultrafilter of B is *-consistent, for if $U = Th(u)$, then $u \in \langle\alpha_u^*\rangle X$ but $u \notin \langle\alpha_u^n\rangle X$ for any n, where X is any clopen set not containing u, therefore $\langle\alpha_u^*\rangle X \in U$ but $\langle\alpha_u^n\rangle X \notin U$ for any n

But the characteristic algebra of any standard Kripke model has *-consistent ultrafilters, namely Th(s) for any state s; this is because $\alpha^* = \alpha^{rtc}$ in standard models, thus

$$s \in \langle\alpha^*\rangle X \text{ iff } \exists n\ s \in \langle\alpha^n\rangle X,$$

or in other words

$$\langle\alpha^*\rangle X \in Th(s) \text{ iff } \exists n\ \langle\alpha^n\rangle X \in Th(s)$$

This completes the proof. □

An open problem

Let $D = (K,B,\langle\rangle)$ be a separable dynamic algebra. The property:

(22) any $Y \in B$, $Y \neq 0$ extends to a *-consistent ultrafilter

is not true is all dynamic algebras; indeed, our counterexample was constructed expressly not to satisfy (22). But our counterexample is uncountable, and any *countable* dynamic algebra satisfies (22). This can be proved easily using the Tarski-Rasiowa-Sikorski Theorem [BS, Theorem 1.4.10], which states that if A_i are countably many subsets of a Boolean algebra, each with a supremum X_i, then any nonzero element Y extends to an ultrafilter U preserving these suprema, in the sense that $X_i \in U$ iff some element of A_i is in U. Thus any Y extends to a *-consistent ultrafilter, since there are only countably many *-consistency conditions (21).

The dual of (22) is the statement

(23) every clopen set contains a *-consistent point

where a **-consistent point* is one not contained in any $\langle\alpha^*\rangle X - \langle\alpha^{rtc}\rangle X$. The dual of the Tarski-Rasiowa-Sikorski Theorem is known as the Baire Category Theorem. It states that, for sufficiently well-behaved spaces (which countable dynamic spaces are), no Y may be the union of countably many nowhere dense sets. Thus if D is countable, since every $\langle\alpha^*\rangle X - \langle\alpha^{rtc}\rangle X$ is nowhere dense, every Y contains a *-consistent point.

These remarks raise the question: is every *countable* separable dynamic algebra isomorphic to $C(A)$ for some standard A? Countable dynamic algebras satisfy certain desirable properties; for example, they are metrizable. Such a result would still have important implications in dynamic logic, since the language is usually taken to be countable.

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