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Church–Rosser Made Easy

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For Jurek, on the Occasion of his Sixtieth Birthday

Abstract. The Church–Rosser theorem states that the λ-calculus is confluent under β-reductions. The standard proof of this result is due to Tait and Martin-Löf. In this note, we present an alternative proof based on the notion of *acceptable orderings*. The technique is easily modified to give confluence of the $\beta\eta$-calculus.

Keywords: lambda-calculus, confluence, Church–Rosser theorem

1. Introduction

A fundamental result in the λ-calculus is *confluence*: if $e \stackrel{*}{\to} e_1$ and $e \stackrel{*}{\to} e_2$ by some arbitrary sequences of reductions, then there exists e_3 such that $e_1 \stackrel{*}{\to} e_3$ and $e_2 \stackrel{*}{\to} e_3$. This is result is originally due to Alonzo Church and J. Barkley Rosser in 1936 [2] and is known as the *Church–Rosser theorem*.

The standard proof of this result, as presented by Barendregt [1], is due to Tait and Martin-Löf. The Tait–Martin-Löf proof is based on an auxiliary reduction relation defined by formal rules and is very amenable to machine verification. Several implementations in automated deduction systems have been reported, along with various improvements and simplifications [5, 6, 7].

Besides the Tait–Martin-Löf proof, Barendregt [1, Chp. 11] presents another proof based on the idea of *developments*. This proof involves tracing a set of occurrences of redexes in a term through a

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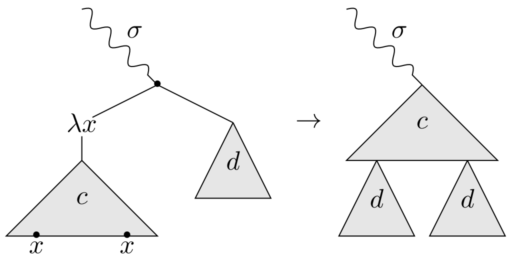

Figure 1: A β-reduction at σ

sequence of reductions. It is somewhat more transparent than the Tait–Martin-Löf proof, but is longer and unfortunately does not readily generalize to the $\beta\eta$-calculus.

The shortest proof along these lines is perhaps the proof of Takahashi [7] (see also [5, 6]). A good exposition is given in [5].

The same result in the presence of η-reductions was first proved by Curry and Feys [3] and later improved and generalized by Hindley [4]. These proofs show that every reduction sequence can be transformed to one in a special normal form.

In this note we offer a short alternative treatment based on the notion of *acceptable orderings*. We prove confluence under β-reductions in Section 3. A slight modification of the proof admits η-reductions as well, and we present this modification in Section 4.

2. Preliminaries

2.1. λ-Terms as Labeled Trees

We view λ-terms as finite labeled trees. Let ω^* denote the set of finite-length strings of natural numbers. A *tree* is a nonempty prefix-closed subset of ω^*. A λ-*term* is a partial function e whose domain $\mathrm{dom}\ e \subseteq \omega^*$ is a finite tree such that $e(\sigma)$ is either an abstraction operator λx, in which case σ has one child $\sigma 0$; the application operator, in which case σ has a left child $\sigma 0$ and a right child $\sigma 1$; or a variable, in which case σ is a leaf.

The *subterm of* e *rooted at* $\sigma \in \mathrm{dom}\ e$ is the term $e \restriction \sigma = \lambda\tau.\, e(\sigma\tau)$ (λ is used here as a meta-operator). Note that if σ is a prefix of τ, then $e \restriction \tau$ is a subterm of $e \restriction \sigma$.

We consider α-equivalent terms identical. The β-reduction rule then takes the following form. Suppose σ is a β-redex in e, say $e \restriction \sigma = (\lambda x.\, c)\, d$. This is replaced at σ by the corresponding contractum consisting of the term c with d substituted for all free occurrences of x, renaming bound variables as necessary to avoid capture (Fig. 1).

2.2. Acceptable Orderings

If σ and τ are both β-redexes in e and σ is a proper prefix of τ, then $e \restriction \tau$ is a proper subterm of $e \restriction \sigma$. If we reduce σ before reducing τ, then τ will in general no longer be a redex; indeed, it may no longer even exist in the resulting tree. However, if we reduce τ first, then σ is still a redex in the resulting tree, although the subterm at σ may have changed.

More generally, if $A \subseteq \operatorname{dom} e$ is a set of β-redexes in e, and we reduce them in some order consistent with the subterm relation—that is, we reduce $\sigma \in A$ only if all proper extensions $\sigma\tau \in A$ have already been reduced—then every redex in A will still be available when it is time to reduce it, and it will be possible to reduce all of them. Moreover, the actual order does not matter, as long as it is consistent with the subterm relation.

Formally, we say that a linear ordering $\sigma_1, \ldots, \sigma_n$ of the elements of A is *acceptable* if $\sigma_i = \sigma_j\tau$ implies $i \leq j$; in other words, the sequence $\sigma_1, \ldots, \sigma_n$ is a subsequence of some total extension of the partial order $\{(\sigma\tau,\, \sigma) \mid \sigma,\ \tau \in \omega^*\}$ (or, if you like, a topological sort of A with respect to the edges $(\sigma\tau,\, \sigma)$).

Acceptable orderings of A are not unique, but this does not matter: it is easily proved inductively that all acceptable orderings give reduction sequences of the same length, namely the cardinality of A, and the resulting final terms are the same up to α-equivalence. Let us call this final term $\theta_A(e)$, as it depends only on e and A and not on the order of reductions.

Takahashi's treatment [7] (see also [5, 6]) is based on the notion of *parallel reduction*, which is derived from the idea of *complete development* of Tait and Martin-Löf (see [1]). In her words, "Intuitively, the property [of confluence] is satisfied by the term e^* which is obtained from e by contracting all β-redexes in e simultaneously." Essentially, e^* corresponds to $\theta_A(e)$, where A is the set of all redexes in e, reduced in some canonical acceptable order. Our treatment gives a little more flexibility in that A can be any set of redexes, not just the set of all redexes, and they can be reduced in any acceptable order. This flexibility will pay off in the subsequent development, especially in the treatment of the $\beta\eta$-calculus in Section 4.

$$
\begin{tikzcd}
& e \arrow[dl, "B"'] \arrow[dr, "A"] & \\
\theta_B(e) \arrow[dr, "A"] & & \theta_A(e) \arrow[dl, "B"'] \\
& \begin{array}{c}\theta_B(\theta_A(e))\\ = \theta_A(\theta_B(e))\end{array} &
\end{tikzcd}
$$

Figure 2

For $\sigma \in \omega^*$, let $\sigma{\downarrow} = \{\sigma\tau \mid \tau \in \omega^*\}$, the set of strings of which σ is a prefix. For $\sigma \in \operatorname{dom} e$, $\sigma{\downarrow} \cap \operatorname{dom} e$ represents the set of (occurrences of) subterms of $e{\upharpoonright}\sigma$.

For $A, X \subseteq \omega^*$, write $A \rhd X$ if there exists σ such that $A \subseteq \sigma{\downarrow}$ and $\sigma{\downarrow} \cap X = \emptyset$. If $A \rhd X$, then A and X are disjoint, and there exists an acceptable ordering of $A \cup X$ in which all elements of A precede all elements of X.

3. Confluence under β-Reductions

We start by proving confluence under β-reductions in some special cases, building up to the general result in Theorem 3.1.

Lemma 3.1. Let A and B be two sets of redexes of e such that all elements of A are prefix-incomparable to all elements of B. The terms $\theta_B(\theta_A(e))$ and $\theta_A(\theta_B(e))$ exist and are equal. This gives the confluent diagram illustrated in Fig. 2.

Proof:
Both $\theta_B(\theta_A(e))$ and $\theta_A(\theta_B(e))$ represent the reduction of the redexes in $A \cup B$ in different acceptable orders, thus both terms are equal to $\theta_{A\cup B}(e)$. $\square$

Lemma 3.2. Let σ be a redex of e and let A be a set of redexes of e such that $A \subseteq \sigma\!\downarrow$. There exists a set B of redexes of $\theta_\sigma(e)$ such that $B \subseteq \sigma\!\downarrow$ and

$$\theta_C(\theta_A(e)) \;=\; \theta_B(\theta_\sigma(e)),$$

where $C = \{\sigma\}$ if $\sigma \notin A$ and $C = \emptyset$ if $\sigma \in A$. This gives the confluent diagram illustrated in Fig. 3.

Proof:
Suppose first that $\sigma \notin A$. Let $e\!\upharpoonright\!\sigma = (\lambda x.\, c)\, d$. The set A may contain redexes in c and d. Reducing σ first, a copy of d replaces each free occurrence of x in c (see Fig. 1). If we then reduce the redexes in

$$\begin{tikzcd}
& e \arrow[dl, "A"'] \arrow[dr, "\sigma"] & \\
\theta_A(e) \arrow[dr, "C"'] & & \theta_\sigma(e) \arrow[dl, "B"] \\
& \theta_C(\theta_A(e)) = \theta_B(\theta_\sigma(e)) &
\end{tikzcd}$$

Figure 3

these copies of d in some acceptable order, then reduce the remaining redexes in c in some acceptable order, this yields the same result as reducing the redexes in d and c in some acceptable order before reducing σ, then reducing σ.

Formally, take

$$B = \{\sigma\gamma_i \mid 1 \le i \le m\} \cup \{\sigma\delta_i\tau_j \mid 1 \le i \le k,\ 1 \le j \le n\},$$

where

$$A = \{\sigma 00\gamma_i \mid 1 \le i \le m\} \cup \{\sigma 1\tau_j \mid 1 \le j \le n\}$$

and the free occurrences of x in c are located at $\{\sigma 00\delta_1, \ldots, \sigma 00\delta_k\}$. The elements of A of the form $\sigma 00\gamma_i$ represent the redexes in c, which after reducing σ become the elements of B of the form $\sigma\gamma_i$. The elements of A of the form $\sigma 1\tau_j$ represent the redexes in d, which after reducing σ become the elements of B of the form $\sigma\delta_i\tau_j$ representing the corresponding redexes in the copies of d that replaced the free occurrences of x in c. In Fig. 1, $k = 2$.

If $\sigma \in A$, then it must appear last in any acceptable ordering of A. By the previous argument, there exists $B \subseteq \sigma\!\downarrow$ such that $\theta_\emptyset(\theta_A(e)) = \theta_\sigma(\theta_{A-\{\sigma\}}(e)) = \theta_B(\theta_\sigma(e))$. $\square$

Lemma 3.3. Let A and X be sets of redexes of e such that $A \triangleright X$. There exists a set B of redexes of $\theta_X(e)$ such that

$$\theta_X(\theta_A(e)) \;=\; \theta_B(\theta_X(e)).$$

Proof:
This follows by induction on the cardinality of X using Lemmas 3.1 and 3.2. Starting with $X_0 = X$ and $B_0 = A$, construct a sequence of sets X_i and B_i by taking the elements of X one at a time in some acceptable order $\tau_0, \ldots, \tau_{n-1}$, maintaining the invariant $B_i \triangleright X_i$. Initially, $B_0 \triangleright X_0$ by assumption, as witnessed by some σ_0. Lemma 3.1 is used if the next $\tau_i \in X$ is prefix-incomparable to all elements of B_i, in which case $B_{i+1} = B_i$, and $\sigma_{i+1} = \sigma_i$ witnesses $B_{i+1} \triangleright X_{i+1}$; and Lemma 3.2 is used if the next element $\tau_i \in X$ is a prefix of all elements of B_i, in which case B_{i+1} is the B of Lemma 3.2, and $\sigma_{i+1} = \tau_i$ witnesses $B_{i+1} \triangleright X_{i+1}$. The set X_{i+1} is $X_i - \{\tau_i\}$. The final set B in the statement of the lemma is B_n.

Fig. 4 illustrates the case $X = \{\tau_0, \tau_1, \tau_2\}$. $\square$

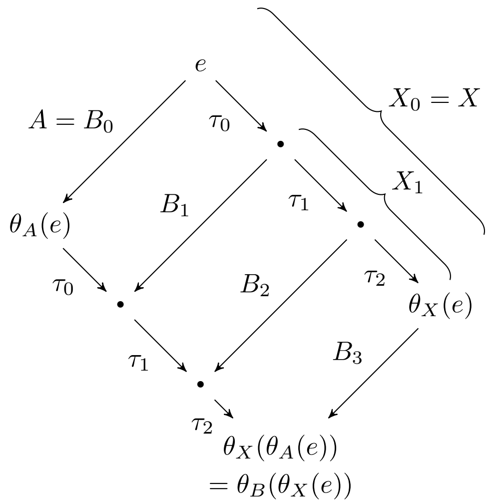

Figure 4

Note that the second case of Lemma 3.2 (in which $C = \emptyset$) was not used in the proof of Lemma 3.3. It will be needed in the proof of the next lemma.

Lemma 3.4. Let A be an arbitrary set of redexes of e, and let σ be a redex of e. There exist redex sets C of $\theta_A(e)$ and B of $\theta_\sigma(e)$ such that

$$\theta_C(\theta_A(e)) \;=\; \theta_B(\theta_\sigma(e)).$$

This gives the confluent diagram of Fig. 3 (the same diagram as for Lemma 3.2, but with a different interpretation of the symbols).

Proof:
Partition A into $A_1 = \sigma{\downarrow} \cap A$ and $A_2 = A - A_1$. Then $A_1 \rhd A_2$. By Lemma 3.2, there exist a set $B_1 \subseteq \sigma{\downarrow}$ of redexes of $\theta_\sigma(e)$ and $C_1 \subseteq \{\sigma\}$ such that

$$\theta_{C_1}(\theta_{A_1}(e)) \;=\; \theta_{B_1}(\theta_\sigma(e)). \tag{1}$$

Take $B = B_1 \cup A_2$. Since $B_1 \subseteq \sigma{\downarrow}$, $C_1 \subseteq \sigma{\downarrow}$, and $\sigma{\downarrow} \cap A_2 = \emptyset$, we have $B_1 \rhd A_2$ and $C_1 \rhd A_2$. By Lemma 3.3, there exists a set C of redexes of $\theta_{A_2}(\theta_{A_1}(e)) = \theta_A(e)$ such that

$$\theta_C(\theta_{A_2}(\theta_{A_1}(e))) \;=\; \theta_{A_2}(\theta_{C_1}(\theta_{A_1}(e))). \tag{2}$$

Then

$$\begin{aligned}
\theta_C(\theta_A(e)) &= \theta_C(\theta_{A_2}(\theta_{A_1}(e))) && \text{since } A_1 \rhd A_2 \\
&= \theta_{A_2}(\theta_{C_1}(\theta_{A_1}(e))) && \text{by (2)} \\
&= \theta_{A_2}(\theta_{B_1}(\theta_\sigma(e))) && \text{by (1)} \\
&= \theta_B(\theta_\sigma(e)) && \text{since } B_1 \rhd A_2.
\end{aligned}$$

□

Lemma 3.5. Let $e \xrightarrow{*} e'$ by some arbitrary sequence of β-reductions, and let A be a set of redexes of e. There exists a set B of redexes of e' such that $\theta_A(e) \xrightarrow{*} \theta_B(e')$.

Proof:
This follows in a straightforward fashion by induction on the length of the reduction sequence $e \xrightarrow{*} e'$ by composing the reductions of Lemma 3.4. □

Theorem 3.1. (Church–Rosser Theorem)
Let $e \xrightarrow{*} e_1$ and $e \xrightarrow{*} e_2$ by some arbitrary sequences of β-reductions. There exists e_3 such that $e_1 \xrightarrow{*} e_3$ and $e_2 \xrightarrow{*} e_3$.

Proof:
Lemma 3.5 gives a confluent diagram for each step in the reduction sequence $e \xrightarrow{*} e_1$, and these can be composed to get a confluent diagram for the entire sequence. □

4. Accommodating η

The η-reduction rule is $\lambda x.cx \to c$, where c contains no free occurrences of x. We show in this section that a minor modification of the argument of Section 3 gives confluence under β- and η-reductions.

The main concern is that due to *overlapping redexes*, it is no longer true in general that any set of redexes $A \subseteq \mathrm{dom}\, e$ can be completely reduced simply by reducing them in acceptable order. There are two problematic situations, as illustrated in Fig. 5.

Consider the configuration of Fig. 5(a). There is a β-redex at the root whose left child is an η-redex. If the η-reduction is performed first, the root is no longer a β-redex in general. However, a key

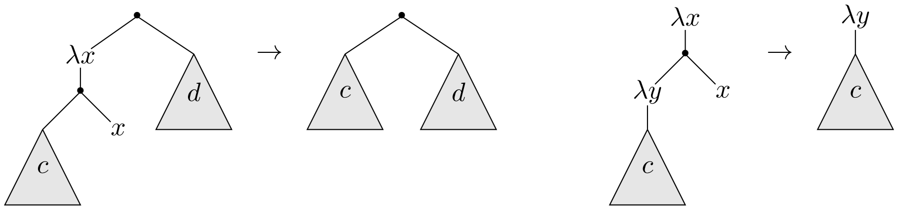

Figure 5: Overlapping redexes: (a) a β-η overlap; (b) an η-β overlap

observation is that we can perform either the β-reduction at the root or the η-reduction at the left child, and the resulting contractum is the same, as shown.

Similarly, Fig. 5(b) shows an η-redex at the root whose only child is a β-redex. As with (a), performing the β-reduction at the child may destroy the η-redex at the root. However, if we perform either reduction, the resulting contractum is the same (up to α-equivalence), as shown.

The solution is simply to disallow redex sets A containing either of these two configurations. Equivalently, A may not contain both σ and $\sigma 0$ for any σ. We will call a redex set $A \subseteq \operatorname{dom} e$ ***overlap-free*** if this property holds. Any overlap-free set of redexes can be fully reduced in acceptable order.

The entire development of Section 3 now goes through with minor modification. The formal statements of Lemmas 3.1–3.5 and Theorem 3.1 are modified as follows:

Lemma 4.1. Let A and B be two overlap-free sets of redexes of e such that all elements of A are prefix-incomparable to all elements of B. The terms $\theta_B(\theta_A(e))$ and $\theta_A(\theta_B(e))$ exist and are equal.

Lemma 4.2. Let σ be a redex of e and let A be an overlap-free set of redexes of e such that $A \subseteq \sigma\!\downarrow$. There exists an overlap-free set B of redexes of $\theta_\sigma(e)$ such that $B \subseteq \sigma\!\downarrow$ and $\theta_C(\theta_A(e)) = \theta_B(\theta_\sigma(e))$, where $C = \{\sigma\}$ if both $\sigma, \sigma 0 \notin A$ and $C = \emptyset$ if either $\sigma \in A$ or $\sigma 0 \in A$.

Lemma 4.3. Let A and X be sets of redexes of e such that $A \rhd X$ and $A \cup X$ is overlap-free. There exists an overlap-free set B of redexes of $\theta_X(e)$ such that $\theta_X(\theta_A(e)) = \theta_B(\theta_X(e))$.

Lemma 4.4. Let A be an arbitrary overlap-free set of redexes of e, and let σ be a redex of e. There exist overlap-free redex sets C of $\theta_A(e)$ and B of $\theta_\sigma(e)$ such that $\theta_C(\theta_A(e)) = \theta_B(\theta_\sigma(e))$.

Lemma 4.5. Let $e \xrightarrow{*} e'$ by some arbitrary sequence of β- and η-reductions, and let A be an overlap-free set of redexes of e. There exists an overlap-free set B of redexes of e' such that $\theta_A(e) \xrightarrow{*} \theta_B(e')$.

Theorem 4.1. (Church–Rosser Theorem for the $\beta\eta$-calculus)
Let $e \xrightarrow{*} e_1$ and $e \xrightarrow{*} e_2$ by some arbitrary sequences of β- and η-reductions. There exists e_3 such that $e_1 \xrightarrow{*} e_3$ and $e_2 \xrightarrow{*} e_3$.

Lemma 4.2 for the case of σ a β-redex is the same as in Lemma 3.2, with the extra observation that B cannot contain overlapping redexes if A did not. For the case of σ an η-redex, if $A = \{\sigma 00\gamma_i \mid 1 \leq$

$i \leq m\}$, we take $B = \{\sigma\gamma_i \mid 1 \leq i \leq m\}$. In both cases, if $\sigma \in A$ or $\sigma 0 \in A$, we can take $C = \emptyset$, otherwise $C = \{\sigma\}$.

For Lemma 4.4, we can assume without loss of generality that $A \cup \{\sigma\}$ is overlap-free; for if $\tau \in A$ and either $\tau = \sigma 0$ or $\sigma = \tau 0$, we can just replace σ with τ in the proof, as $\theta_\sigma(e) = \theta_\tau(e)$. We can then conclude that the $B_1 \cup A_2$ and $C_1 \cup A_2$ constructed in the proof are overlap-free. All else is the same as in Lemma 3.4.

The proofs of Lemmas 4.1, 4.3, 4.5, and Theorem 4.1 go through essentially unchanged from Lemmas 3.1, 3.3, 3.5, and Theorem 3.1, respectively.

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