Principled Programming

Introduction to Coding in Any Imperative Language

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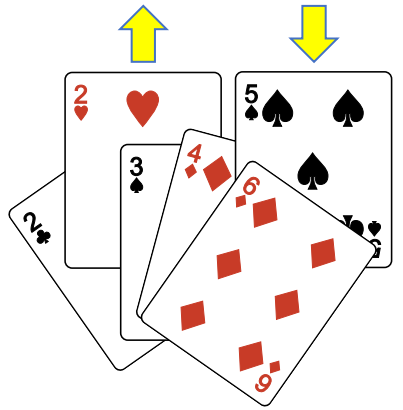
One-Dimensional Array Rearrangements

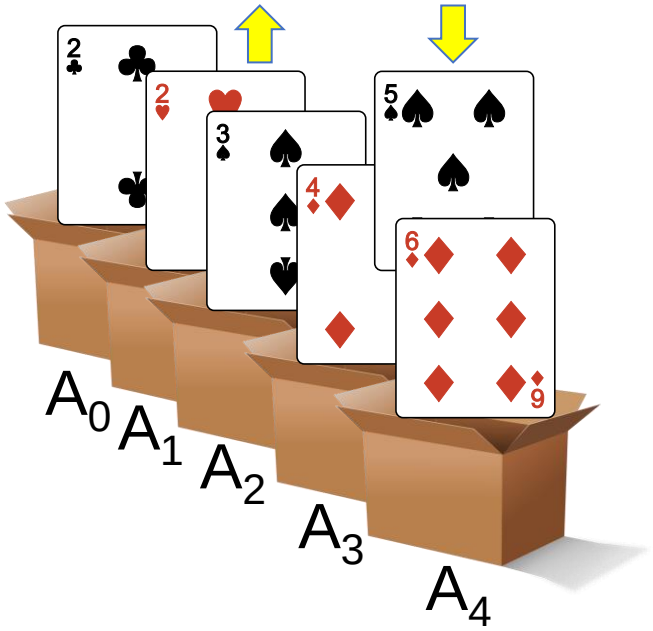
The need to rearrange values in an array is commonplace, and facility in doing so is important.

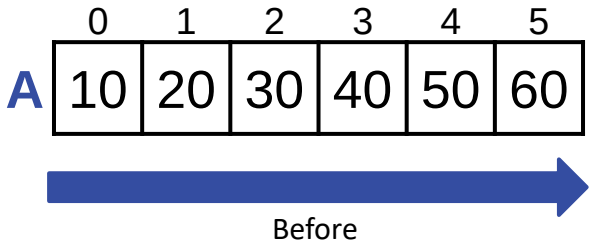
Everyday experience is helpful, e.g., manipulating a hand of playing cards. However, beware that when cards are deleted or inserted, others move over automagically. A better analogy is cards in boxes, but even this is flawed because values are *copied* from variables, not *pulled*, like cards.

We consider:

- Reverse
- LeftShift
- LeftRotate
- Partitioning
- Collation







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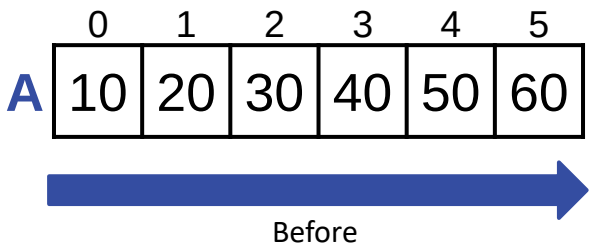

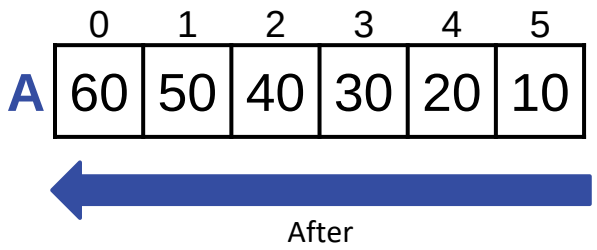

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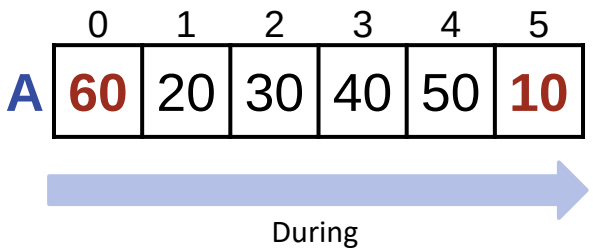

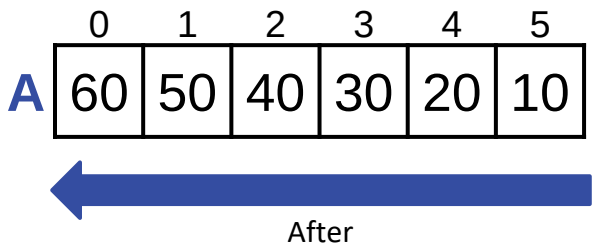

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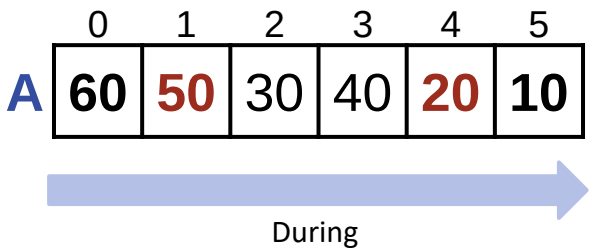

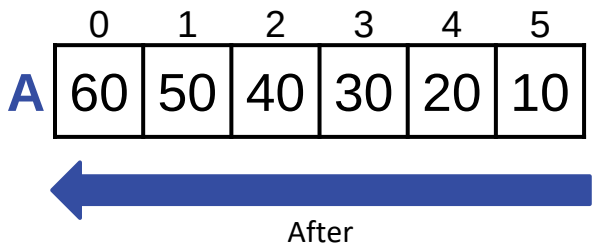

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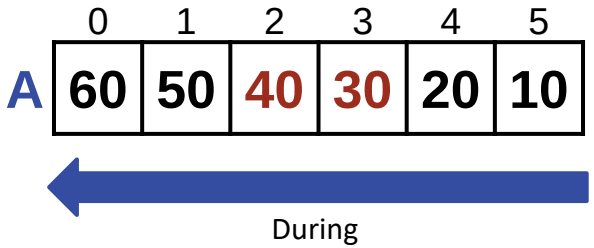

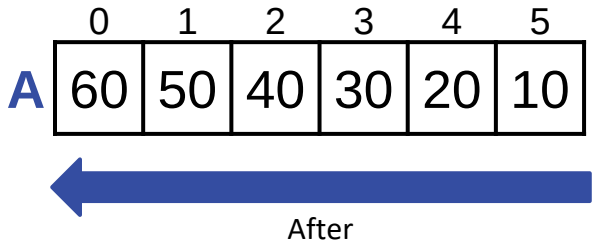

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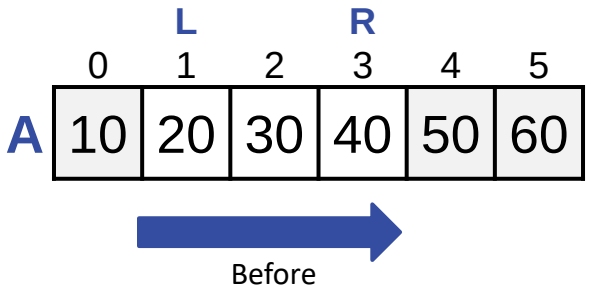

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/* Given int array A[0..n-1], reverse the order of the subsequence A[L..R]
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static void Reverse( int A[], int L, int R ) {
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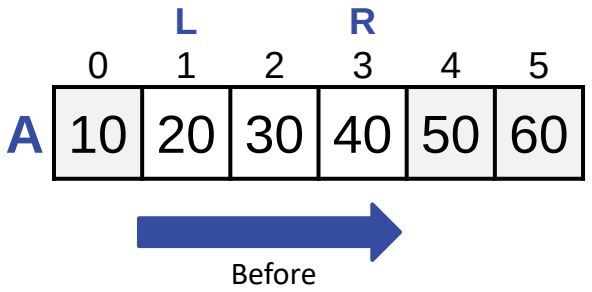

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static void Reverse( int A[], int L, int R ) {
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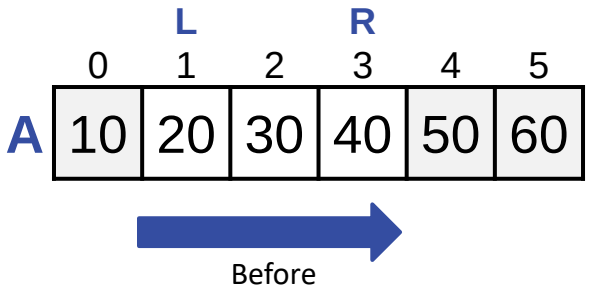

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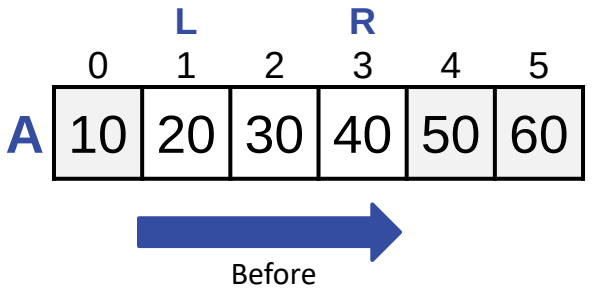
☞ **A statement-comment is written as a statement in a high-level language, e.g., English. As such, it is a specification for code not yet written.**



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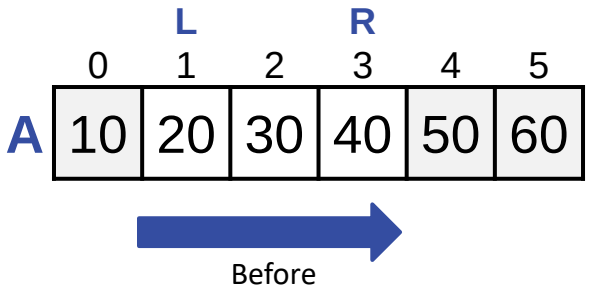

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But when the time comes, “you gotta do what you gotta do”.

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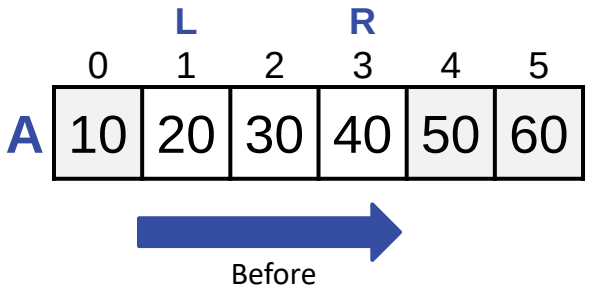

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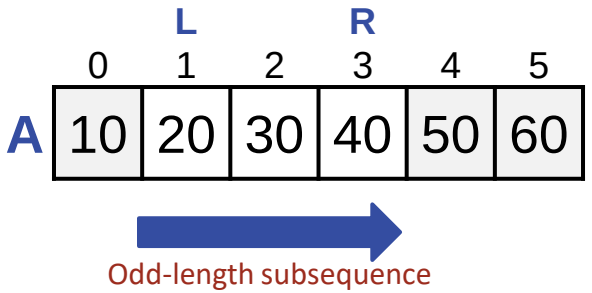
☞ **Be alert to high-risk coding steps associated with binary choices.**



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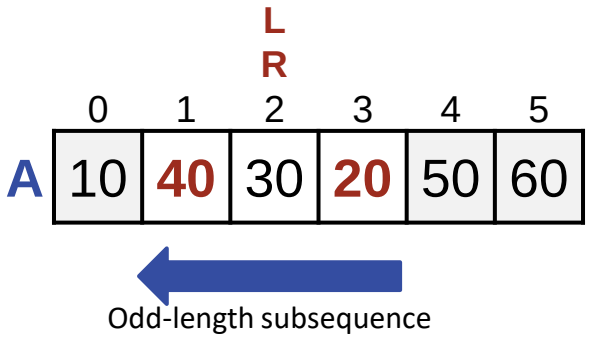

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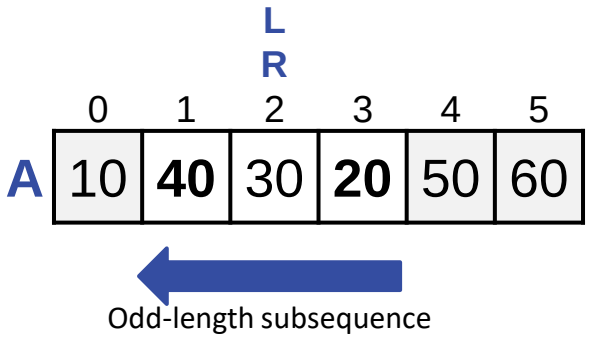

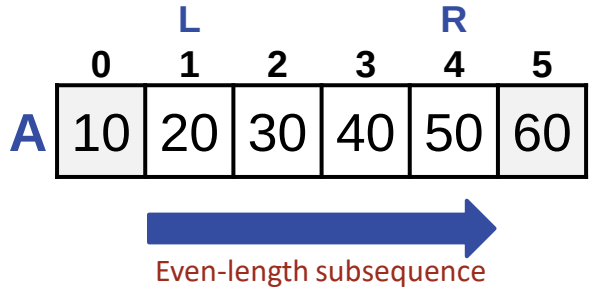

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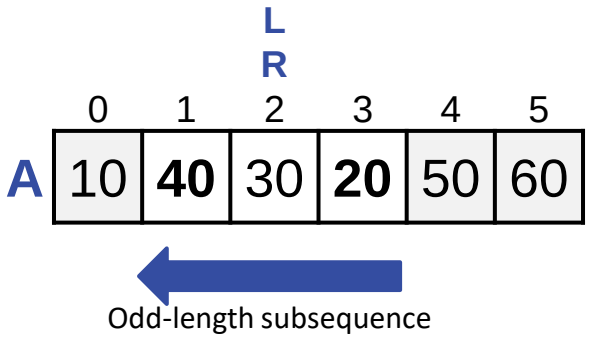

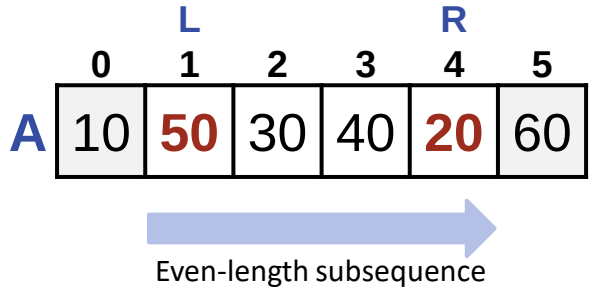

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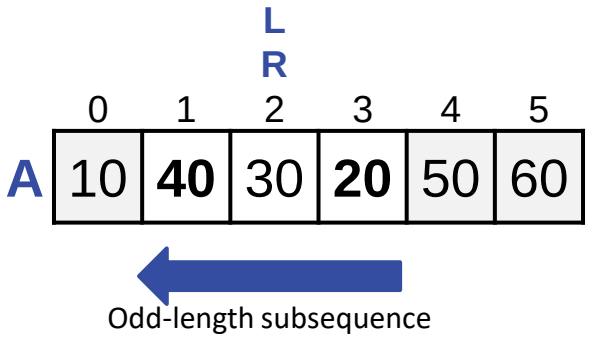

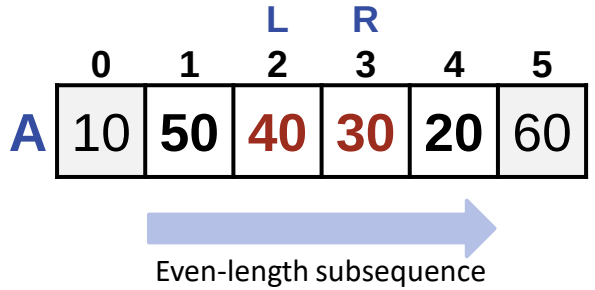

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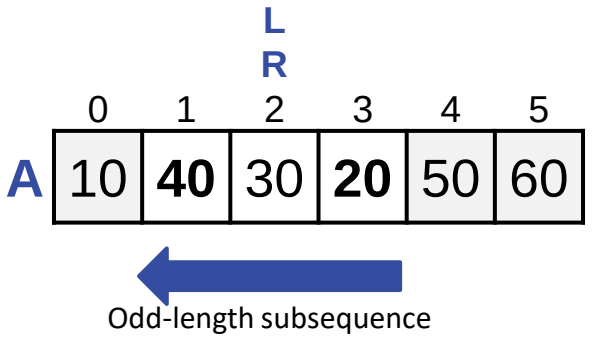

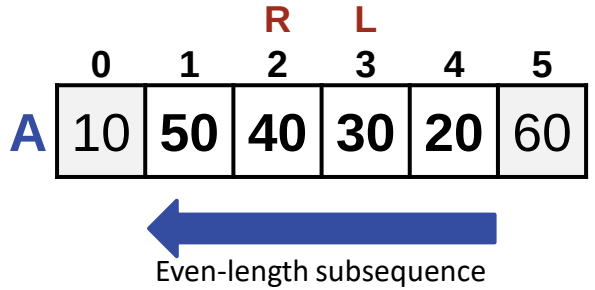

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```

It would have been an error for the *condition* to have been L!=R.

☞ **Be alert to high-risk coding steps associated with binary choices.**

Pure top-down programming is an aspirational goal.
Sometimes, you are so "into the problem in your head" that you find no need to specify first.
You can often slip in something useful after the fact if you follow the "relentless copy editing" precept.

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         int temp = A[L];  A[L] = A[R];  A[R] = temp;
      /* Reduce the size of the not-yet-reversed subsequence by 2. */
         L++; R--;
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   } /* Reverse */
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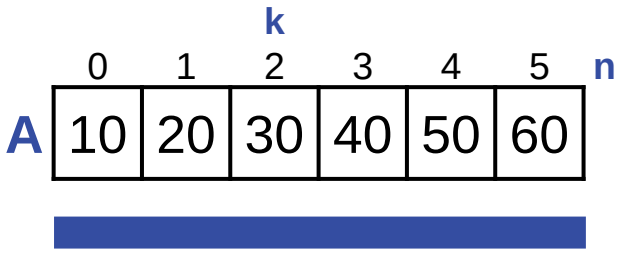
☞ **Repeatedly improve comments by relentless copy editing.**



New Application: Shift an array left k places.

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/* Given array A[0..n-1], and 0≤k, shift elements of A left k places.
   Values shifted off the left end of the array are lost. Values not
   overwritten remain as they were originally. */
static void LeftShiftK( int[] A, int n, int k ) {
   } /* LeftShiftK */
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   for (int j=0; _____; j++) A[j] = A[j+k];
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- ☞ **If you "smell a loop", write it down.**
- ☞ **Decide first whether an iteration is indeterminate (use while) or determinate (use for).**

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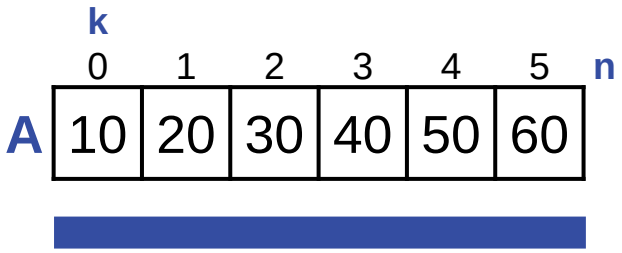
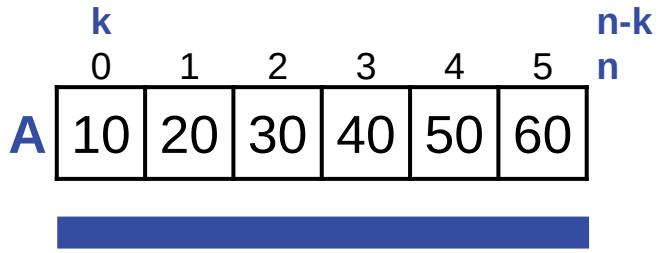
☞ **Boundary conditions. Dead last, but don't forget them.**





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   if ( k>0 )
      for (int j=0; j<n-k; j++) A[j] = A[j+k];
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It would have been correct without this test, but offensive that for k==0 we would do the most work.

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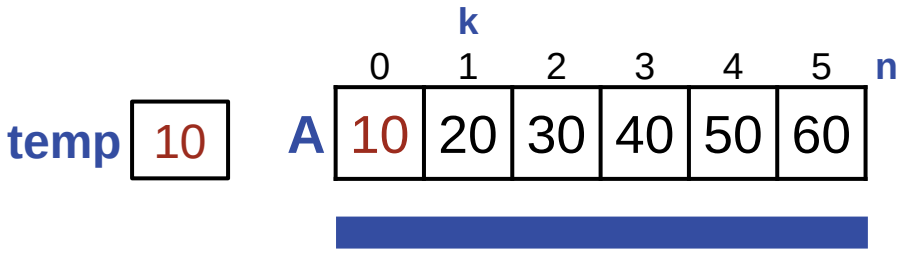
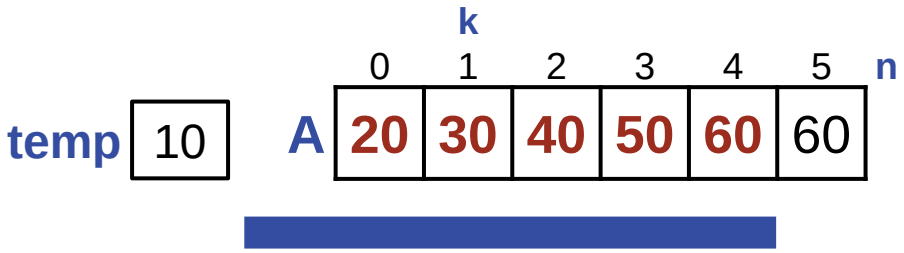


New application: Rotate an array left 1 place.

```
/* Given int array A[0..n-1], shift A[1..n-1] left one place, with the value
   that was originally in A[0] reentering at the right in A[n-1]. */
static void LeftRotateOne(int A[], int n) {
   int temp = A[0];
   LeftShiftK(A, n, 1);
   A[n-1] = temp;
   } /* LeftRotateOne */
```



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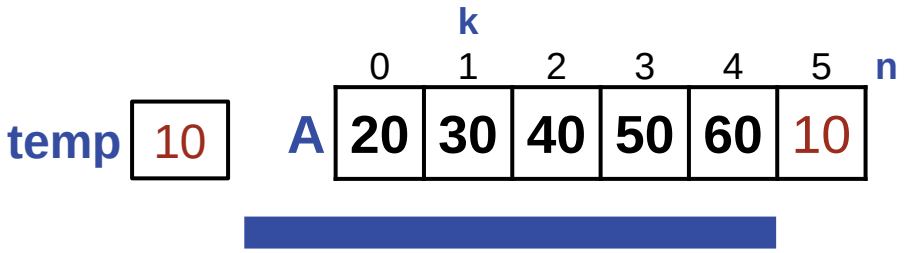

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New Application: Rotate an array left k places.

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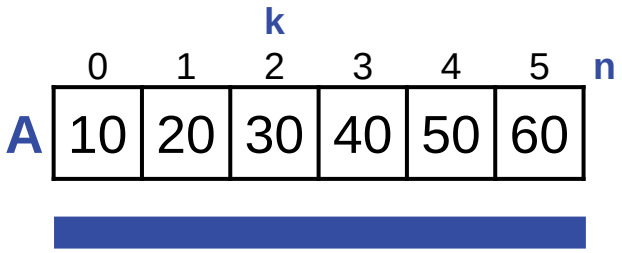
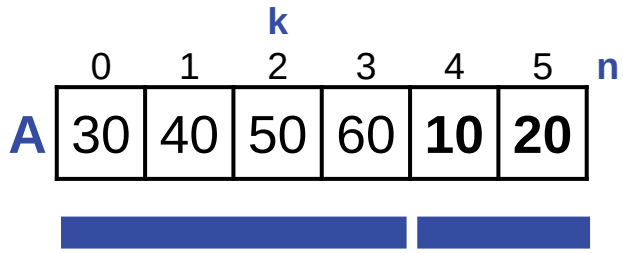
/* Given int array A[0..n-1], and integer k, 0≤k<n, left shift A[k..n-1]
   k places, with values originally in A[0..k-1] reentering at right. */
```





Application: Rotate an array left k places.

```
/* Given int array A[0..n-1], and integer k, 0≤k<n, left shift A[k..n-1]
   k places, with values originally in A[0..k-1] reentering at right. */
```

Application: Rotate an array left k places.

```
/* Given int array A[0..n-1], and integer k, 0≤k<n, left shift A[k..n-1]
   k places, with values originally in A[0..k-1] reentering at right. */
```

We shall consider four distinct approaches:

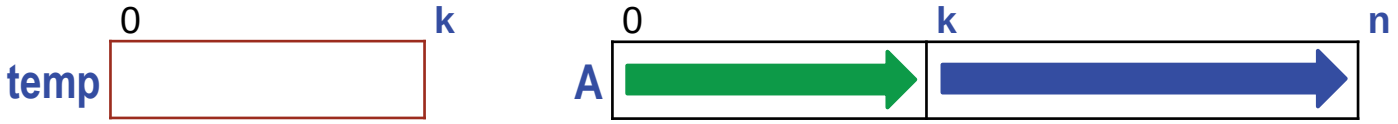
- Repeated Left-Rotate-1
- Swap Generalization
- Three Flips
- Juggle in Cycles

Approach 1: Repeated left rotation 1 place.

```
/* Given int array A[0..n-1], and integer k, 0≤k<n, left shift A[k..n-1]
   k places, with values originally in A[0..k-1] reentering at right. */
   for (int j=0; j<k; j++) LeftRotateOne(A, n);
```

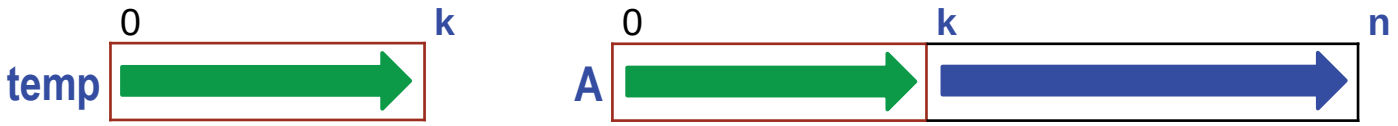
Approach 2: k-wide generalization of swap.

```
/* Given int array A[0..n-1], and integer k, 0≤k<n, left shift A[k..n-1]
   k places, with values originally in A[0..k-1] reentering at right. */
```



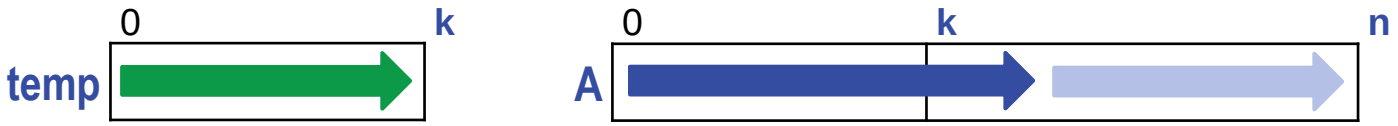
Approach 2: k-wide generalization of swap.

```
/* Given int array A[0..n-1], and integer k, 0≤k<n, left shift A[k..n-1]
   k places, with values originally in A[0..k-1] reentering at right. */
   int temp[] = new int[k];
```



Approach 2: k-wide generalization of swap.

```
/* Given int array A[0..n-1], and integer k, 0≤k<n, left shift A[k..n-1]
   k places, with values originally in A[0..k-1] reentering at right. */
   int temp[] = new int[k];
   /* temp[0..k-1] = A[0..k-1]; */
```



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```
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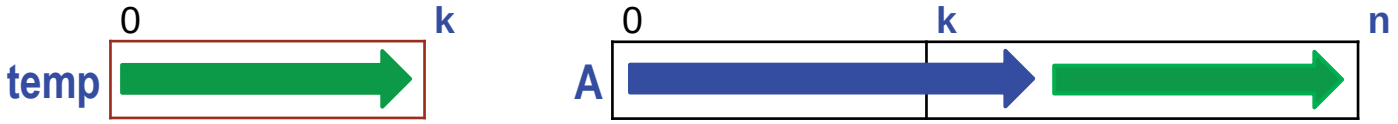
   k places, with values originally in A[0..k-1] reentering at right. */
   int temp[] = new int[k];
   /* temp[0..k-1] = A[0..k-1]; */
   LeftShiftK(A, n, k);
```



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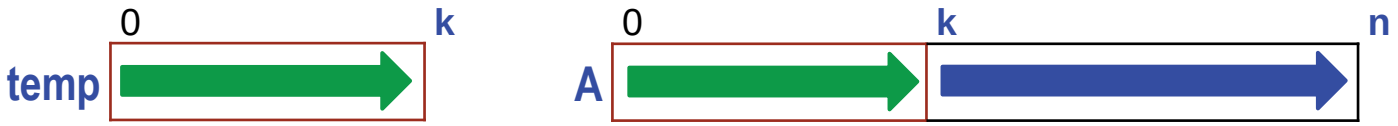
   int temp[] = new int[k];
   /* temp[0..k-1] = A[0..k-1]; */
   LeftShiftK(A, n, k);
   /* A[___..n-1] = temp[0..k-1]; */
```

☞ **Defer challenging code for later; do the easy parts first.**



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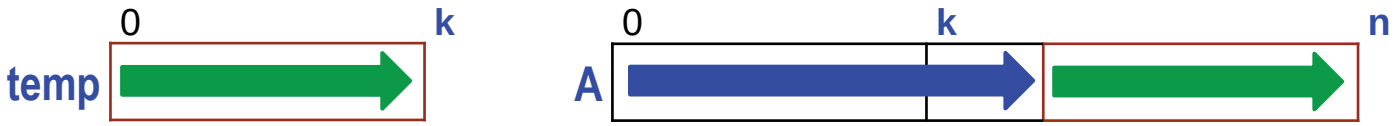
   /* temp[0..k-1] = A[0..k-1]; */
      for (int j=0; j<k; j++) temp[j] = A[j];
   LeftShiftK(A, n, k);
   /* A[___..n-1] = temp[0..k-1]; */
```



Approach 2: k-wide generalization of swap.

```
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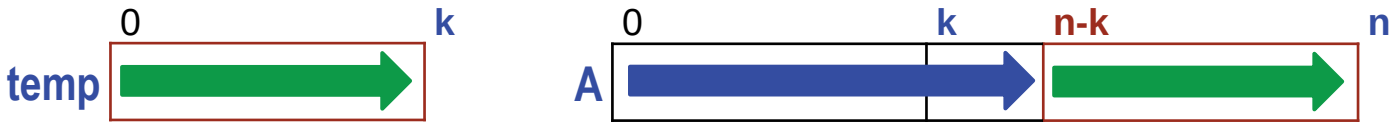
   k places, with values originally in A[0..k-1] reentering at right. */
   int temp[] = new int[k];
   /* temp[0..k-1] = A[0..k-1]; */
      for (int j=0; j<k; j++) temp[j] = A[j];
   LeftShiftK(A, n, k);
   /* A[___..n-1] = temp[0..k-1]; */
      for (int j=0; j<k; j++) A[_____] = temp[j];
```

☞ **Avoid gratuitous differences in code. Reuse code patterns, if possible.**



Approach 2: k-wide generalization of swap.

```
/* Given int array A[0..n-1], and integer k, 0≤k<n, left shift A[k..n-1]
   k places, with values originally in A[0..k-1] reentering at right. */
   int temp[] = new int[k];
   /* temp[0..k-1] = A[0..k-1]; */
      for (int j=0; j<k; j++) temp[j] = A[j];
   LeftShiftK(A, n, k);
   /* A[n-k..n-1] = temp[0..k-1]; */
      for (int j=0; j<k; j++) A[n-k+j] = temp[j];
```

Approach 3: Three Flips. Consider the two parts of the array.

```
/* Given int array A[0..n-1], and integer k, 0≤k<n, left shift A[k..n-1]
   k places, with values originally in A[0..k-1] reentering at right. */
```

Approach 3: Represent the values in those parts as **green** and **blue** arrows.

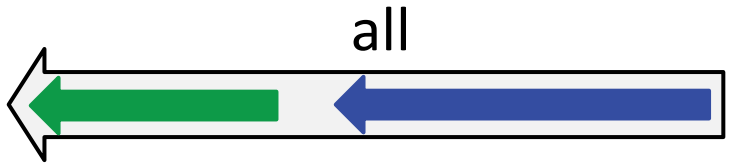
```
/* Given int array A[0..n-1], and integer k, 0≤k<n, left shift A[k..n-1]
   k places, with values originally in A[0..k-1] reentering at right. */
```

Approach 3: Reverse first k

```
/* Given int array A[0..n-1], and integer k, 0≤k<n, left shift A[k..n-1]
   k places, with values originally in A[0..k-1] reentering at right. */
   Reverse(A, 0, k-1);
```

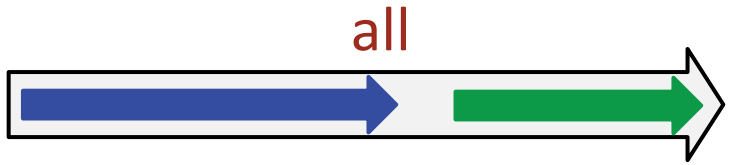
Approach 3: Reverse first k, then rest of elements

```
/* Given int array A[0..n-1], and integer k, 0≤k<n, left shift A[k..n-1]
   k places, with values originally in A[0..k-1] reentering at right. */
   Reverse(A, 0, k-1);
   Reverse(A, k, n-1);
```



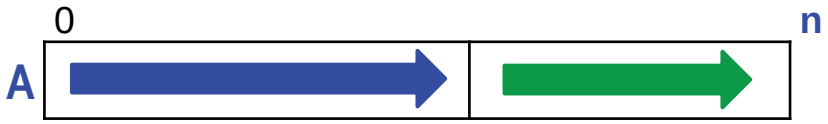
Approach 3: Reverse first k, then rest of elements

```
/* Given int array A[0..n-1], and integer k, 0≤k<n, left shift A[k..n-1]
   k places, with values originally in A[0..k-1] reentering at right. */
   Reverse(A, 0, k-1);
   Reverse(A, k, n-1);
```



Approach 3: Reverse first k, then rest of elements, then all elements.

```
/* Given int array A[0..n-1], and integer k, 0≤k<n, left shift A[k..n-1]
   k places, with values originally in A[0..k-1] reentering at right. */
   Reverse(A, 0, k-1);
   Reverse(A, k, n-1);
   Reverse(A, 0, n-1);
```



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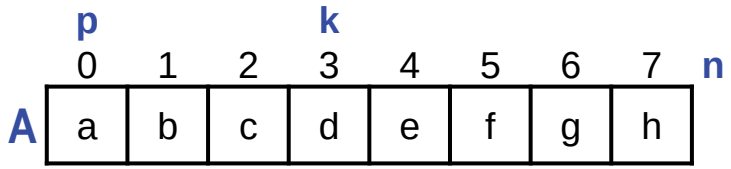
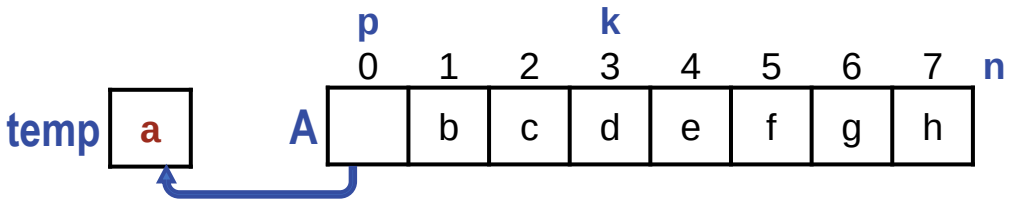
   k places, with values originally in A[0..k-1] reentering at right. */
   Reverse(A, 0, k-1);
   Reverse(A, k, n-1);
   Reverse(A, 0, n-1);
```



Approach 4: Juggle elements in a stride of k.

```
/* Given int array A[0..n-1], and integer k, 0≤k<n, left shift A[k..n-1]
   k places, with values originally in A[0..k-1] reentering at right. */
   int p = 0;                  // Start at A[0]
```



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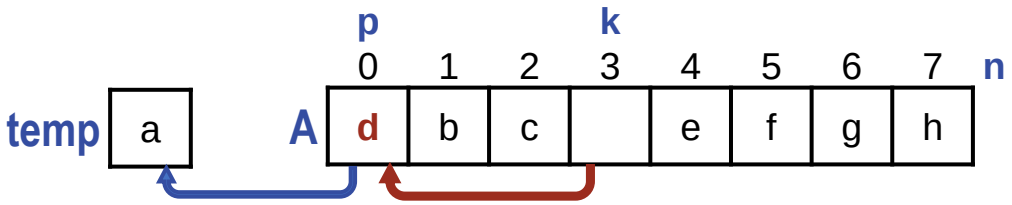
   int p = 0;                  // Start at A[0]
   int temp = A[0];            // and make a hole there.
```



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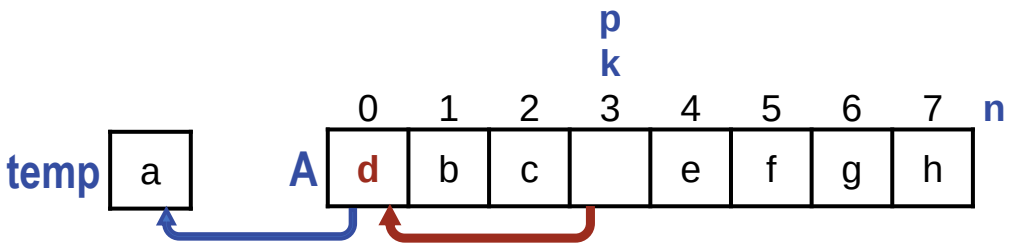
   int p = 0;                 // Start at A[0]
   int temp = A[0];           // and make a hole there.
   while ( __________ ) {
      A[p] = A[  p+k  ];  // Fill hole at p, making a new hole.

      }
```



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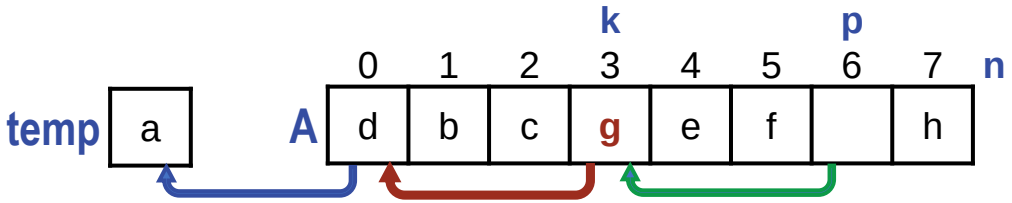
      p = p+k;            // Advance to the new hole.
      }
```



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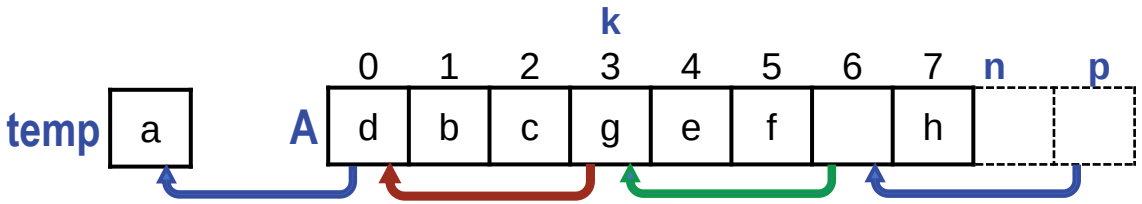
      p = p+k;            // Advance to the new hole.
      }
```



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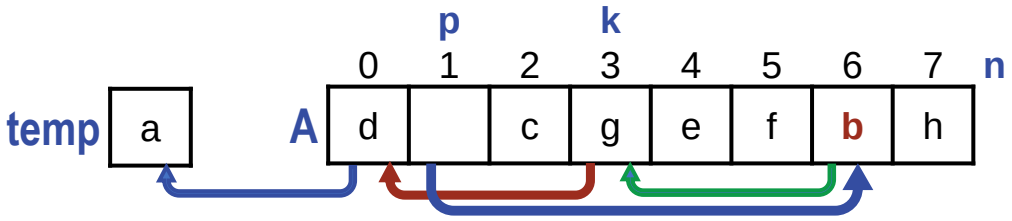
```



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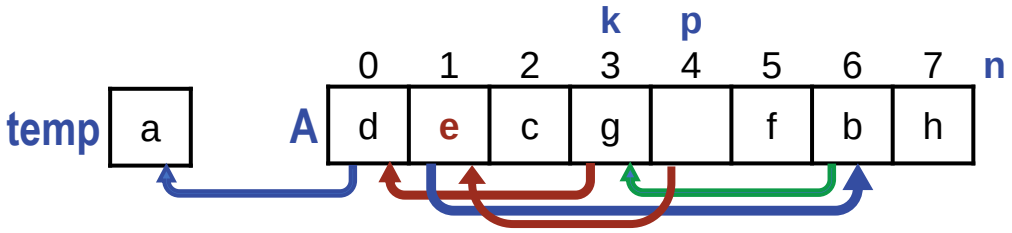
   while ( __________ ) {
      A[p] = A[(p+k)%n];      // Fill hole at p, making a new hole.
      p = (p+k)%n;            // Advance to the new hole.
      }
```



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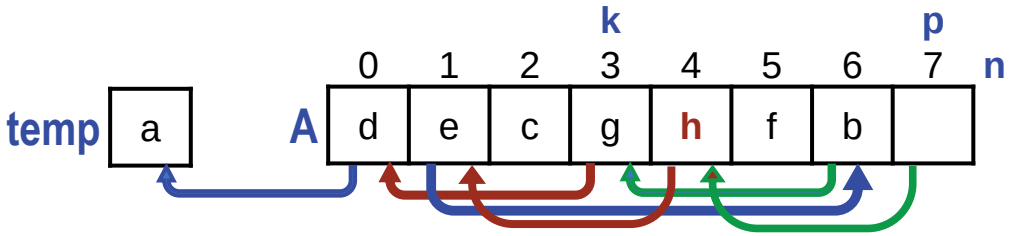
```



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      p = (p+k)%n;         // Advance to the new hole.
      }
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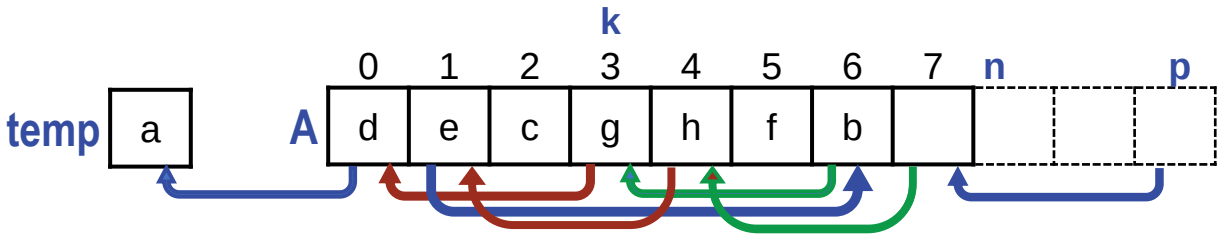

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      p = (p+k)%n;            // Advance to the new hole.
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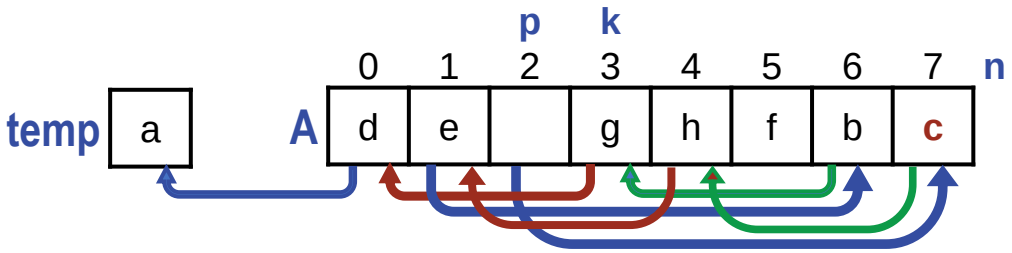

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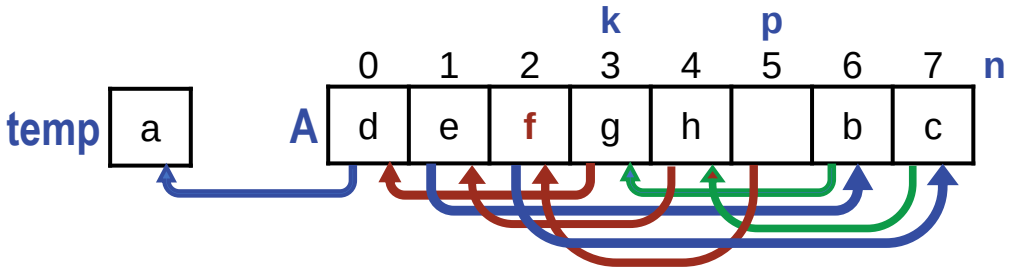

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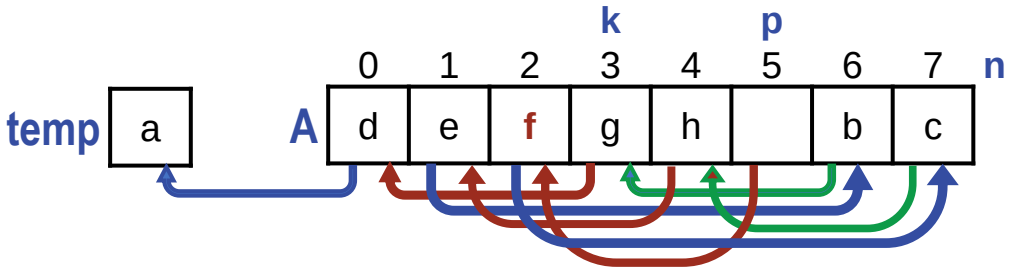
      p = (p+k)%n;        // Advance to the new hole.
      }
```



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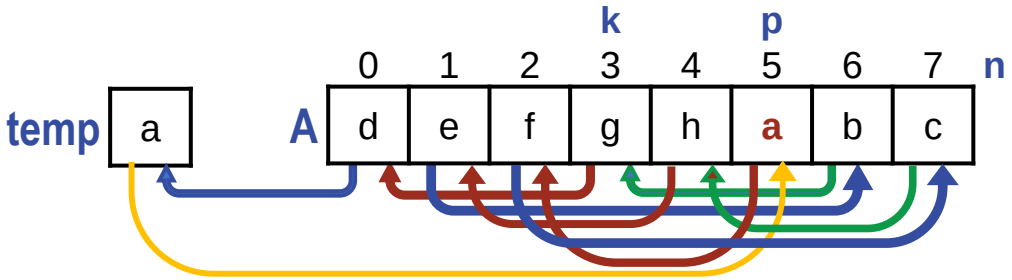

Approach 4: Juggle elements in a stride of k.

```
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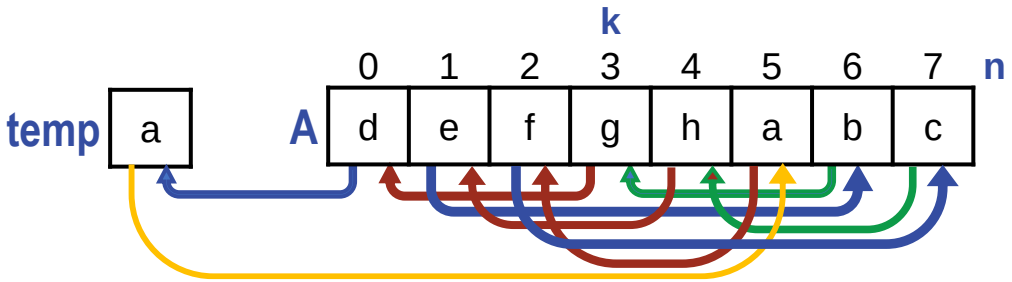
   int p = 0;                 // Start at A[0]
   int temp = A[0];           // and make a hole there.
   while ( (p+k)%n!=0 ) {     // Stop if p is about to be 0 again.
      A[p] = A[(p+k)%n];      // Fill hole at p, making a new hole.
      p = (p+k)%n;            // Advance to the new hole.
      }
   A[p] = temp;               // Fill the last hole from temp.
```



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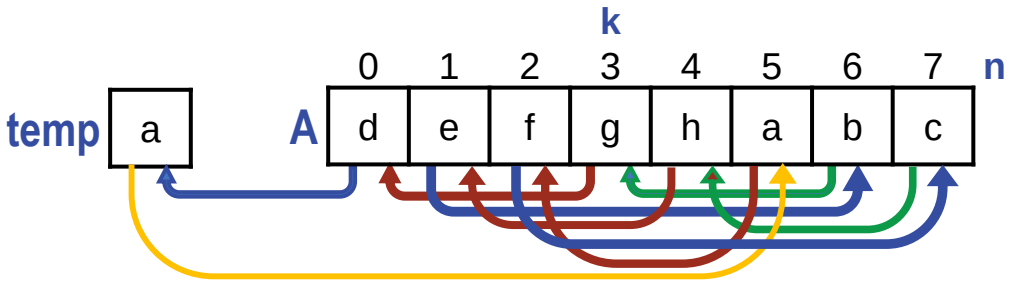
   int temp = A[0];           // and make a hole there.
   while ( (p+k)%n!=0 ) {     // Stop if p is about to be 0 again.
      A[p] = A[(p+k)%n];      // Fill hole at p, making a new hole.
      p = (p+k)%n;            // Advance to the new hole.
      }
   A[p] = temp;               // Fill the last hole from temp.
```

Are we done?

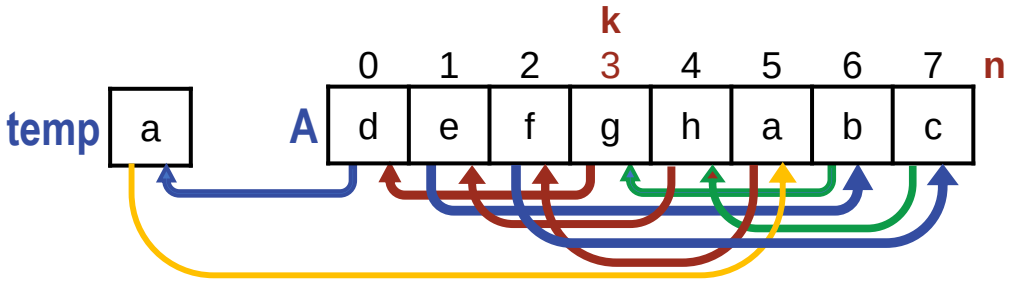


Approach 4: Juggle elements in a stride of k.

```
/* Given int array A[0..n-1], and integer k, 0≤k<n, left shift A[k..n-1]
   k places, with values originally in A[0..k-1] reentering at right. */
   int p = 0;                 // Start at A[0]
   int temp = A[0];           // and make a hole there.
   while ( (p+k)%n!=0 ) {     // Stop if p is about to be 0 again.
      A[p] = A[(p+k)%n];      // Fill hole at p, making a new hole.
      p = (p+k)%n;            // Advance to the new hole.
      }
   A[p] = temp;               // Fill the last hole from temp.
```

☞ **Beware of premature self-satisfaction.**

Are we done?



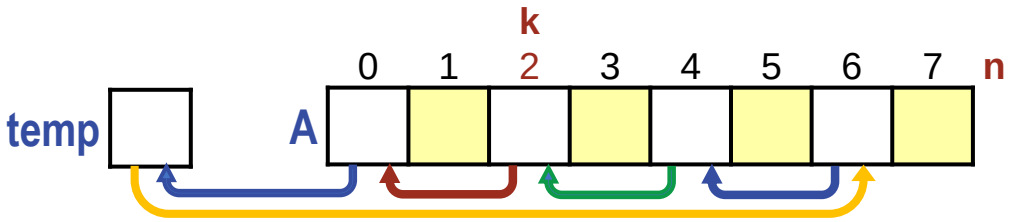
k	n
3	8

Approach 4: Juggle elements in a stride of k.

```
/* Given int array A[0..n-1], and integer k, 0≤k<n, left shift A[k..n-1]
   k places, with values originally in A[0..k-1] reentering at right. */
   int p = 0;                  // Start at A[0]
   int temp = A[0];            // and make a hole there.
   while ( (p+k)%n!=0 ) {      // Stop if p is about to be 0 again.
      A[p] = A[(p+k)%n];       // Fill hole at p, making a new hole.
      p = (p+k)%n;             // Advance to the new hole.
      }
   A[p] = temp;                // Fill the last hole from temp.
```

☞ **Validate output thoroughly.**

Are we done?



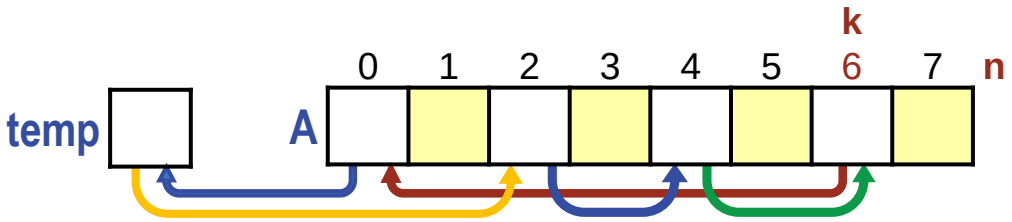
k	n	
3	8	✓
2	8	✗

Approach 4: Juggle elements in a stride of k.

```
/* Given int array A[0..n-1], and integer k, 0≤k<n, left shift A[k..n-1]
   k places, with values originally in A[0..k-1] reentering at right. */
   int p = 0;                 // Start at A[0]
   int temp = A[0];           // and make a hole there.
   while ( (p+k)%n!=0 ) {     // Stop if p is about to be 0 again.
      A[p] = A[(p+k)%n];      // Fill hole at p, making a new hole.
      p = (p+k)%n;            // Advance to the new hole.
      }
   A[p] = temp;               // Fill the last hole from temp.
```

☞ **Validate output thoroughly.**

Are we done? Hardly.



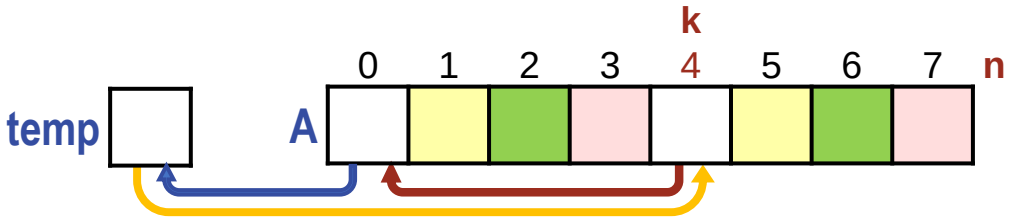
k	n	
3	8	✓
2	8	✗
6	8	✗

Approach 4: Juggle elements in a stride of k.

```
/* Given int array A[0..n-1], and integer k, 0≤k<n, left shift A[k..n-1]
   k places, with values originally in A[0..k-1] reentering at right. */
   int p = 0;                 // Start at A[0]
   int temp = A[0];           // and make a hole there.
   while ( (p+k)%n!=0 ) {     // Stop if p is about to be 0 again.
      A[p] = A[(p+k)%n];      // Fill hole at p, making a new hole.
      p = (p+k)%n;            // Advance to the new hole.
      }
   A[p] = temp;               // Fill the last hole from temp.
```

☞ **Validate output thoroughly.**

Are we done? Hardly.



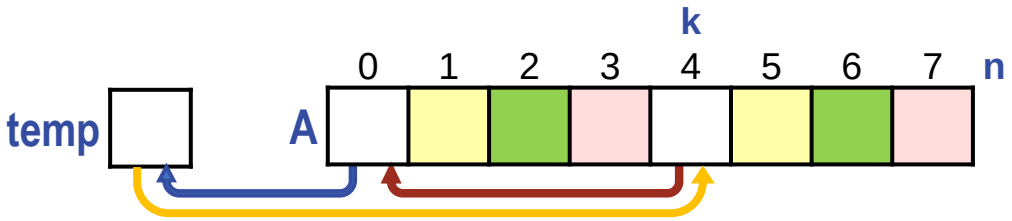
k	n	
3	8	✓
2	8	✗
6	8	✗
4	8	✗

Approach 4: Juggle elements in a stride of k.

```
/* Given int array A[0..n-1], and integer k, 0≤k<n, left shift A[k..n-1]
   k places, with values originally in A[0..k-1] reentering at right. */
   int p = 0;                 // Start at A[0]
   int temp = A[0];           // and make a hole there.
   while ( (p+k)%n!=0 ) {     // Stop if p is about to be 0 again.
      A[p] = A[(p+k)%n];      // Fill hole at p, making a new hole.
      p = (p+k)%n;            // Advance to the new hole.
      }
   A[p] = temp;               // Fill the last hole from temp.
```

☞ **Validate output thoroughly.**

Are we done? Hardly. It only works if k and n are relatively prime!



k	n	
3	8	✓
2	8	✗
6	8	✗
4	8	✗

Approach 4: Juggle elements in a stride of k.

```
/* Given int array A[0..n-1], and integer k, 0≤k<n, left shift A[k..n-1]
   k places, with values originally in A[0..k-1] reentering at right. */
   int p = 0;                 // Start at A[0]
   int temp = A[0];           // and make a hole there.
   while ( (p+k)%n!=0 ) {     // Stop if p is about to be 0 again.
      A[p] = A[(p+k)%n];      // Fill hole at p, making a new hole.
      p = (p+k)%n;            // Advance to the new hole.
      }
   A[p] = temp;               // Fill the last hole from temp.
```

☞ **Validate output thoroughly.**

Are we done? Hardly. It only works if k and n are relatively prime! Now what?

Abandon, or learn that each of `A[0..gcd(k,n)-1]` begins a disjoint cycle.

Approach 4: Juggle elements in a stride of k.

```
/* Given int array A[0..n-1], and integer k, 0≤k<n, left shift A[k..n-1]
   k places, with values originally in A[0..k-1] reentering at right. */
   int g = gcd(n,k);
   for (int j=0; j<g; j++) {
      int p = j;                // Start at A[j]
      int temp = A[p];          // and make a hole there.
      while ( (p+k)%n!=j ) { // Stop if p is about to be j again.
         A[p] = A[(p+k)%n];  // Fill hole at p, making a new hole.
         p = (p+k)%n;           // Advance to the new hole.
         }
      A[p] = temp; // Fill the last hole from temp.
      }
```

Assessment:

Version of Left-Rotate-k	#moves	Explanation
☞ Repeated Left-Rotate-1	$k \cdot n$	Each Left-Rotate-1 moves all n elements. Done k times.
Swap Generalization	$n+k$	The copies into and out from `temp` do $2 \cdot k$ moves, and the shift does $n-k$ moves.
Three Flips	$2 \cdot n$	Each element moves once during the 1st two reverses, and then again for the 3rd reverse.
Juggle in Cycles	$n+\gcd(n,k)$	Each element moves once, plus the first element of each of the gcd(n,k) cycles must first be saved in temp.

Worst #moves, by far. Easiest to understand.

Assessment:

Version of Left-Rotate-k	#moves	Explanation
Repeated Left-Rotate-1	$k \cdot n$	Each Left-Rotate-1 moves all n elements. Done k times.
☞ Swap Generalization	$n+k$	The copies into and out from `temp` do $2 \cdot k$ moves, and the shift does $n-k$ moves.
Three Flips	$2 \cdot n$	Each element moves once during the 1st two reverses, and then again for the 3rd reverse.
Juggle in Cycles	$n+\gcd(n,k)$	Each element moves once, plus the first element of each of the gcd(n,k) cycles must first be saved in temp.

Reasonable #moves but not *in situ*, i.e., requires extra space for `temp`.

Assessment:

Version of Left-Rotate-k	#moves	Explanation
Repeated Left-Rotate-1	$k \cdot n$	Each Left-Rotate-1 moves all n elements. Done k times.
Swap Generalization	$n+k$	The copies into and out from `temp` do $2 \cdot k$ moves, and the shift does $n-k$ moves.
☞ Three Flips	$2 \cdot n$	Each element moves once during the 1st two reverses, and then again for the 3rd reverse.
Juggle in Cycles	$n+\gcd(n,k)$	Each element moves once, plus the first element of each of the gcd(n,k) cycles must first be saved in temp.

Reasonable #moves. Good locality.

Assessment:

Version of Left-Rotate-k	#moves	Explanation
Repeated Left-Rotate-1	$k \cdot n$	Each Left-Rotate-1 moves all n elements. Done k times.
Swap Generalization	$n+k$	The copies into and out from `temp` do $2 \cdot k$ moves, and the shift does $n-k$ moves.
Three Flips	$2 \cdot n$	Each element moves once during the 1st two reverses, and then again for the 3rd reverse.
☞ Juggle in Cycles	$n+\gcd(n,k)$	Each element moves once, plus the first element of each of the gcd(n,k) cycles must first be saved in temp.

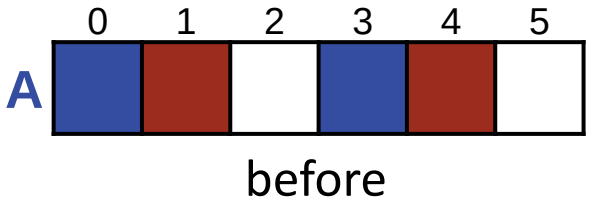
Hardest to understand. Poor locality.

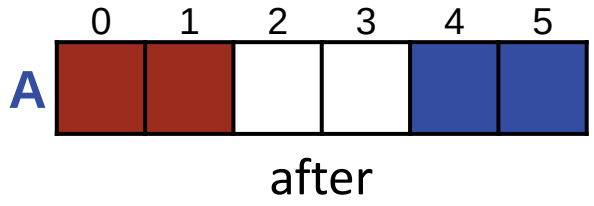
Assessment:

Version of Left-Rotate-k	#moves	Explanation
Repeated Left-Rotate-1	$k \cdot n$	Each Left-Rotate-1 moves all n elements. Done k times.
Swap Generalization	n+k	The copies into and out from `temp` do $2 \cdot k$ moves, and the shift does $n-k$ moves.
✓ Three Flips	$2 \cdot n$	Each element moves once during the 1st two reverses, and then again for the 3rd reverse.
Juggle in Cycles	n+gcd(n,k)	Each element moves once, plus the first element of each of the gcd(n,k) cycles must first be saved in temp.

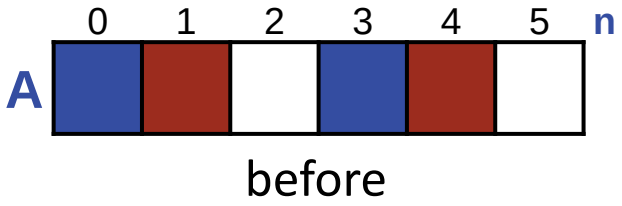
Personal favorite, and really elegant!

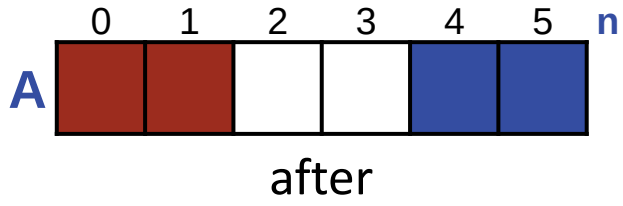
New Application: The Dutch National Flag problem.





Application: Rearrange an array into all red, then all white, then all blue.

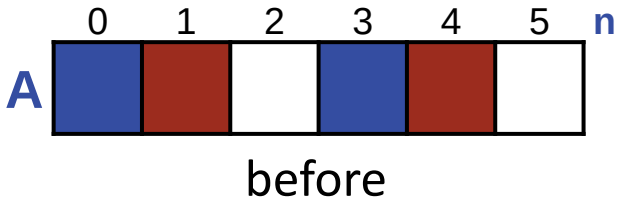


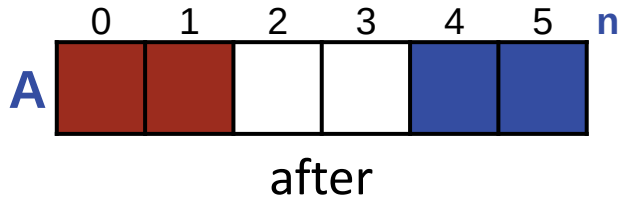


Application: Rearrange an array into all red, then all white, then all blue.

```
/* Given array A[0..n-1] consisting of only three values (red, white,
   and blue), rearrange A into all red, then white, then blue. */
```

☞ **A statement-comment says exactly what code must accomplish, not how it does so.**

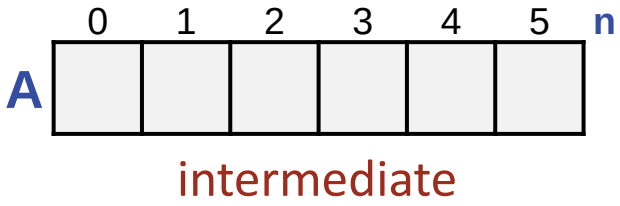


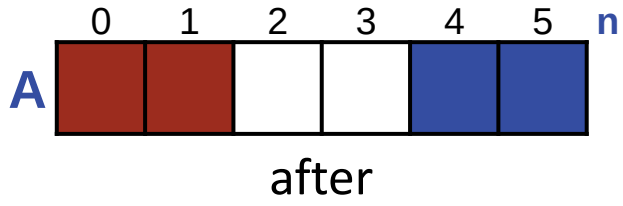


Application: Rearrange an array into all red, then all white, then all blue.

```
/* Given array A[0..n-1] consisting of only three values (red, white,
   and blue), rearrange A into all red, then white, then blue. */
   while ( ______ ) __________
```

☞ **If you "smell a loop", write it down.**



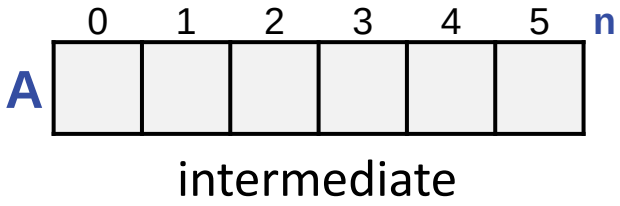


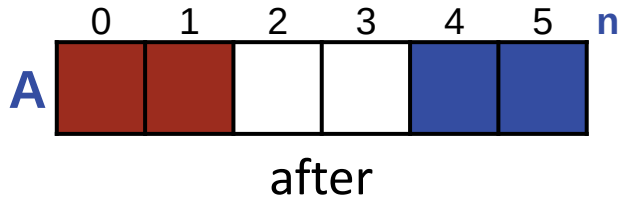
POST

Application: Rearrange an array into all red, then all white, then all blue.

```
/* Given array A[0..n-1] consisting of only three values (red, white,
   and blue), rearrange A into all red, then white, then blue. */
   while ( ______ ) _________
```

☞ **To get to POST iteratively, choose a weakened POST as INVARIANT.**

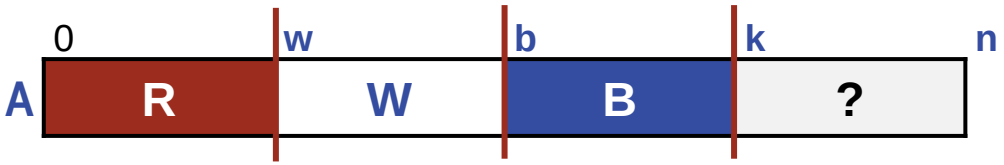


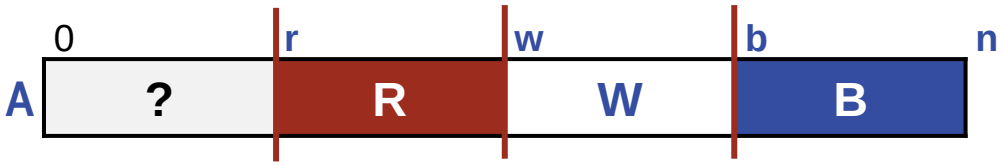


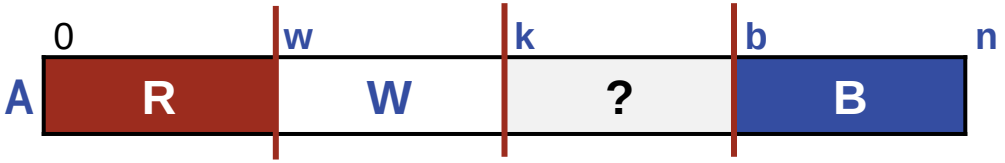
POST

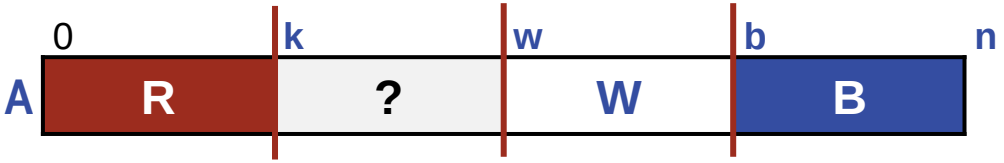
Application: Rearrange an array into all red, then all white, then all blue.

Here are four choices for a weakened POST:

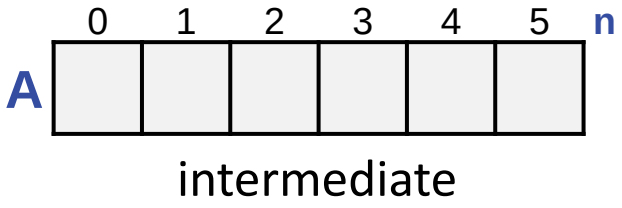


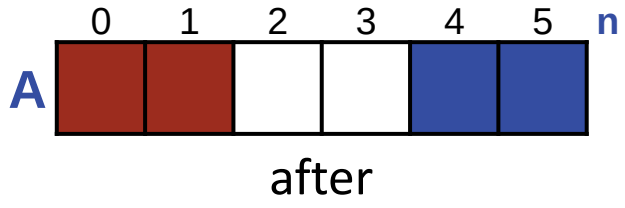






☞ **To get to POST iteratively, choose a weakened POST as INVARIANT.**

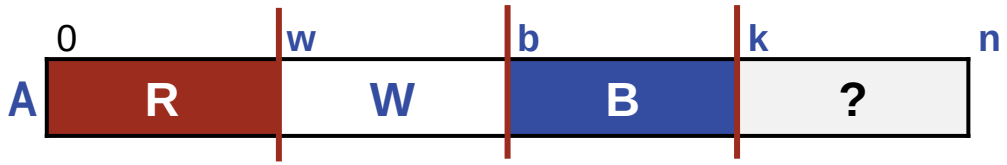


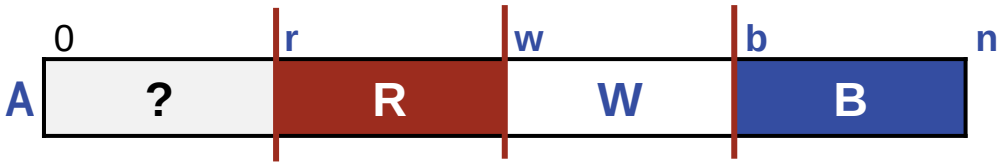


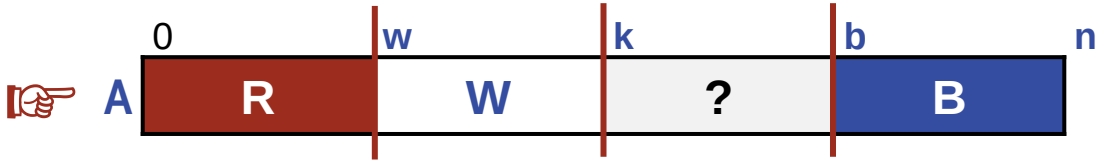
POST

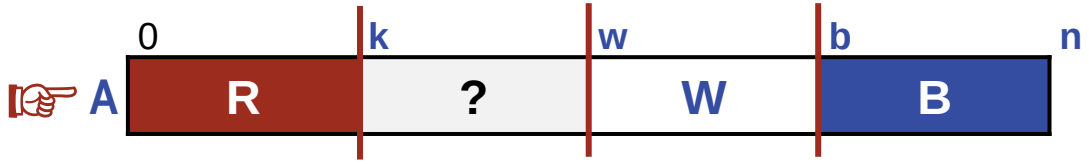
Application: Rearrange an array into all red, then all white, then all blue.

Here are four choices for a weakened POST: How shall we choose?

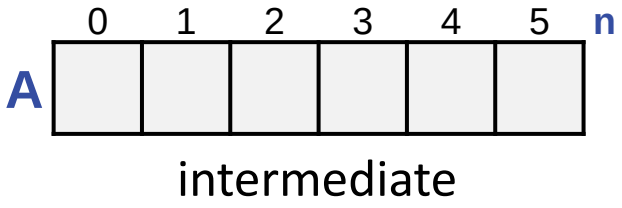


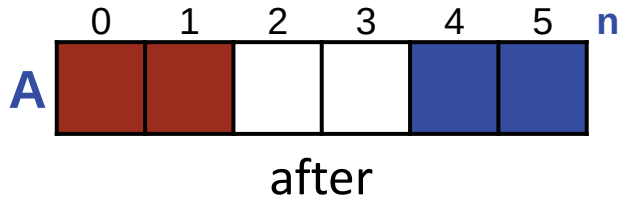






Symmetric, so discard one arbitrarily.

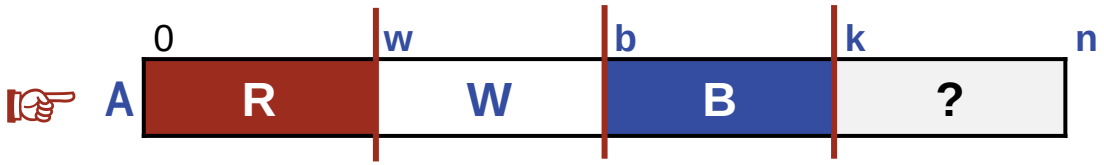


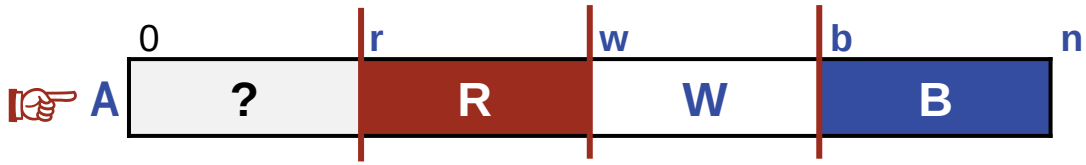


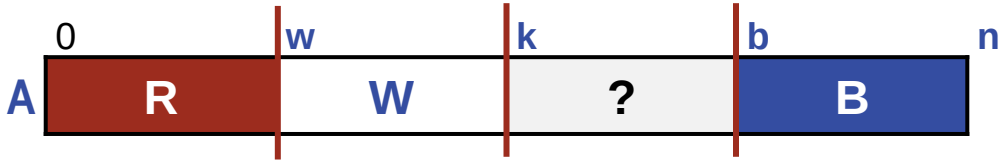
POST

Application: Rearrange an array into all red, then all white, then all blue.

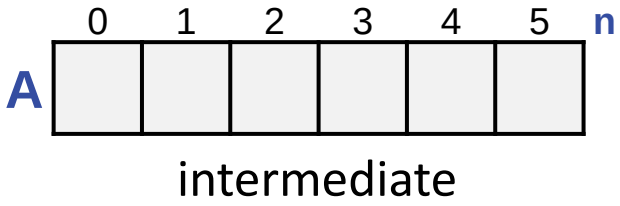
Here are four choices for a weakened POST: How shall we choose?

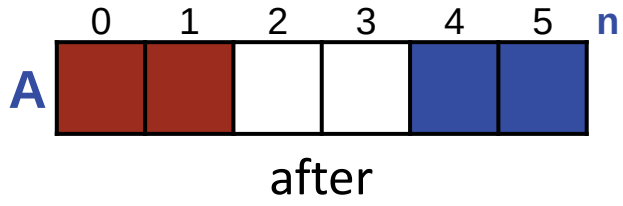






Left more intuitive, because the ? region seems more familiar, so discard right.

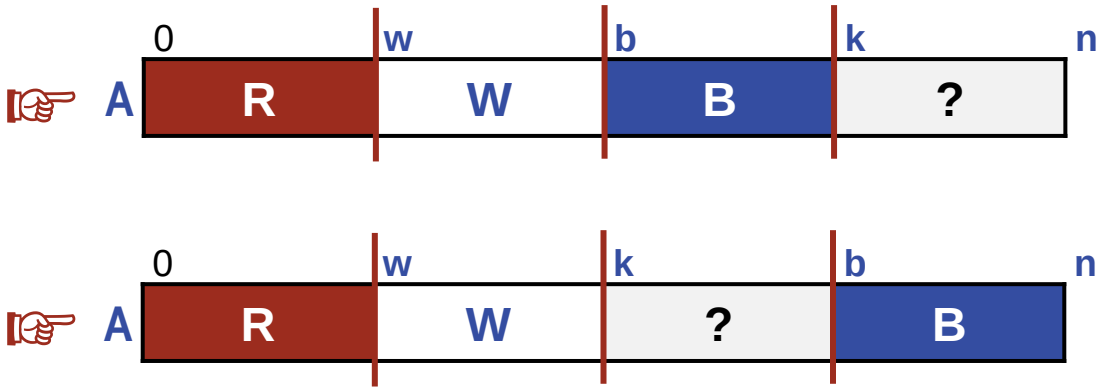




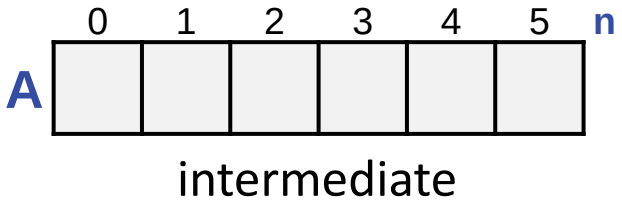
POST

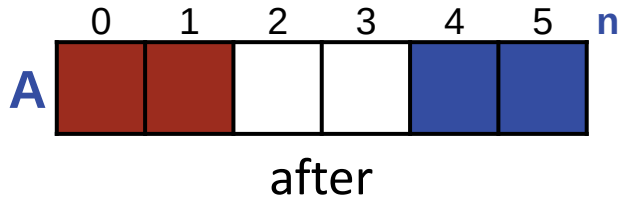
Application: Rearrange an array into all red, then all white, then all blue.

Here are four choices: How shall we choose?



? region of top has only one degree of freedom, but bottom has two. Discard top.

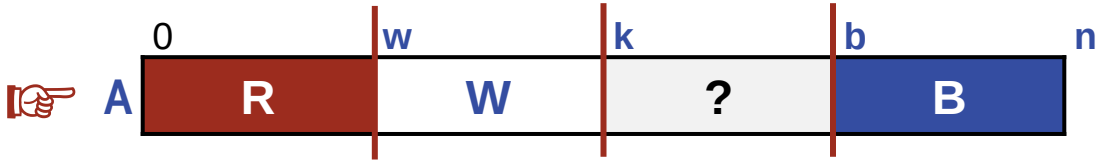




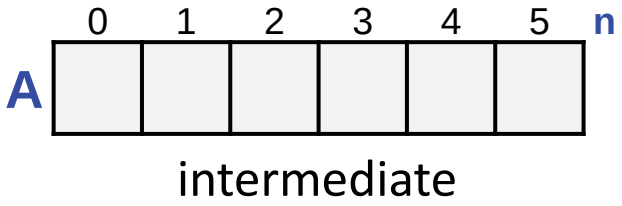
POST

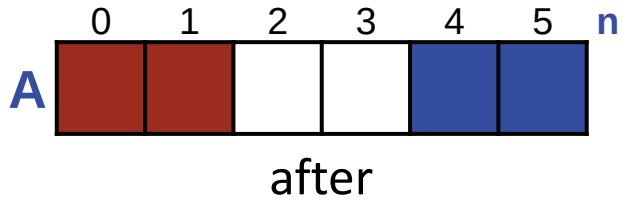
Application: Rearrange an array into all red, then all white, then all blue.

Here are four choices: How shall we choose?



This will be our **INVARIANT**.





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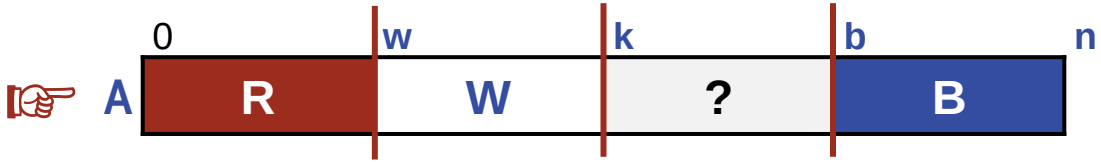
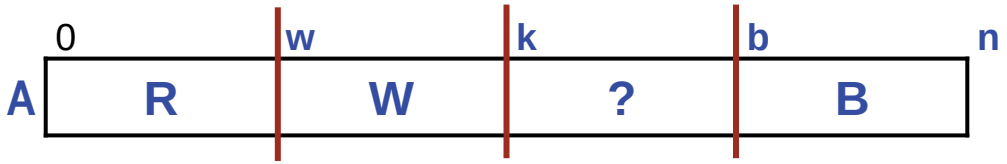


This will be our **INVARIANT**.

We have illustrated that program design and programming can be driven by consideration of the different possible invariants you can think of. One might call this "invariant-driven programming". In this mode of programming, the invariant comes first, not as an afterthought to justify a loop you have already written.



VARIANT: b-k

Application: Rearrange an array into all red, then all white, then all blue.

```
/* Given array A[0..n-1] consisting of only three values (red, white,
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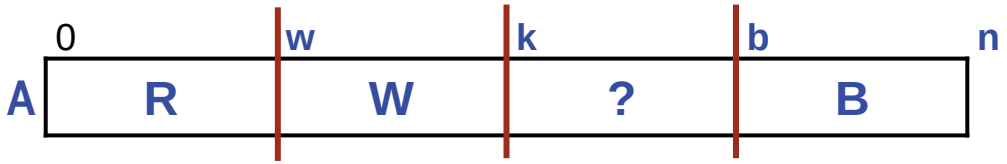
   /* INVARIANT: A[0..w-1] red, A[w..k-1] white, A[b..n-1] blue, for 0≤w≤k≤b≤n. */

   int k = ____; int w = ____; int b = ____;
   while ( ______ ) __________
```



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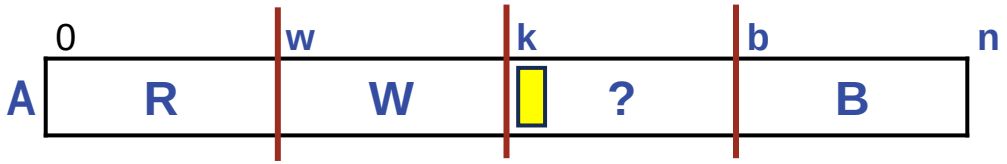
```

☞ **A Case Analysis in the loop body is often needed for characterizing different ways in which to decrease the loop variant while maintaining the loop invariant.**



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/* Given array A[0..n-1] consisting of only three values (red, white,
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      if ( A[k]==______ ) __________
      else if ( A[k] == ______ ) __________
      else __________
```

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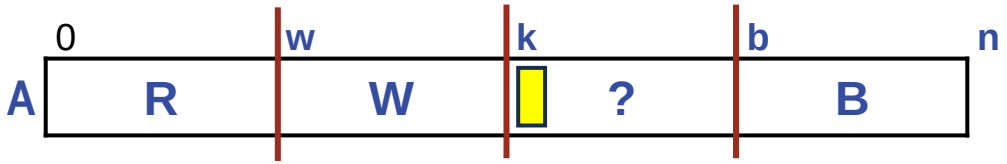

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   while ( ______ )
      if ( A[k]==B ) __________
      else if ( A[k]==R ) __________
      else __________  /* A[k]==W */
```

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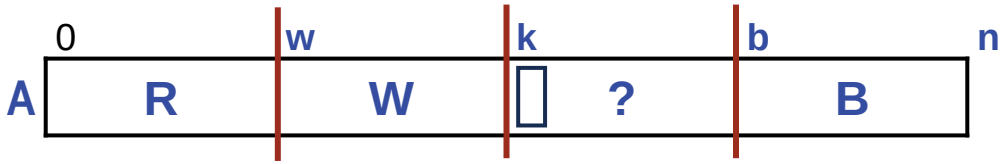

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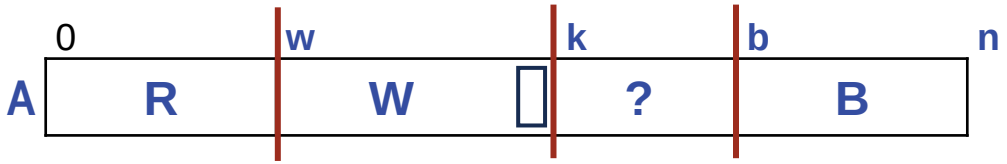

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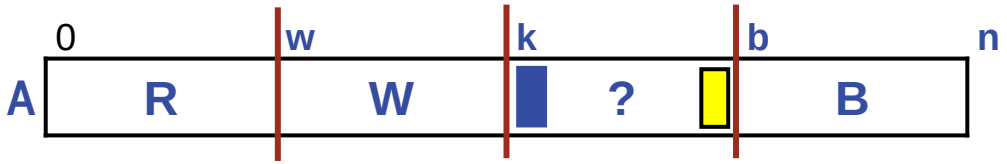

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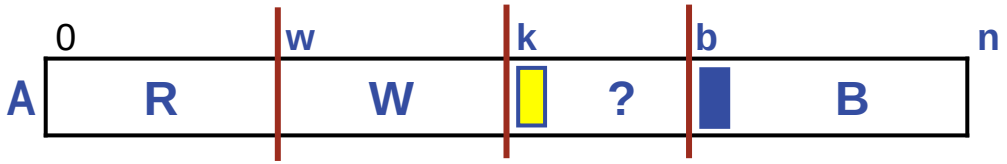

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   int k = ____; int w = ____; int b = ____;
   while ( ______ )
      if ( A[k]==B ) {
         /* Swap A[b-1] and A[k]. */
         b--;
         }
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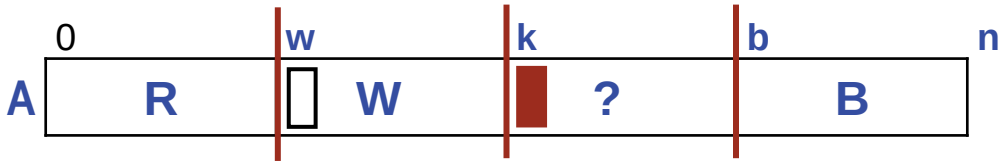

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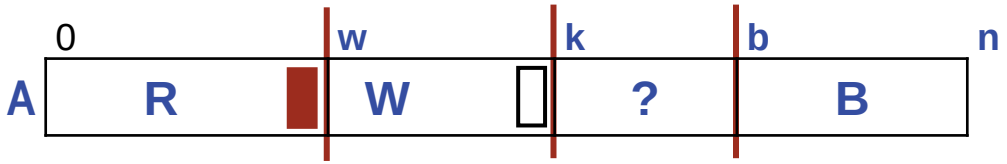

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         b--;
         }
      else if ( A[k]==R ) {
         /* Swap A[w] and A[k]. */
         w++; k++;
         }
      else k++;  /* A[k]==W */
```

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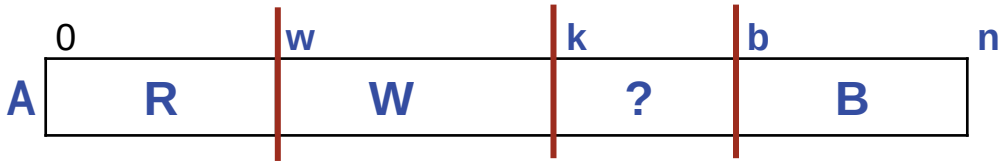

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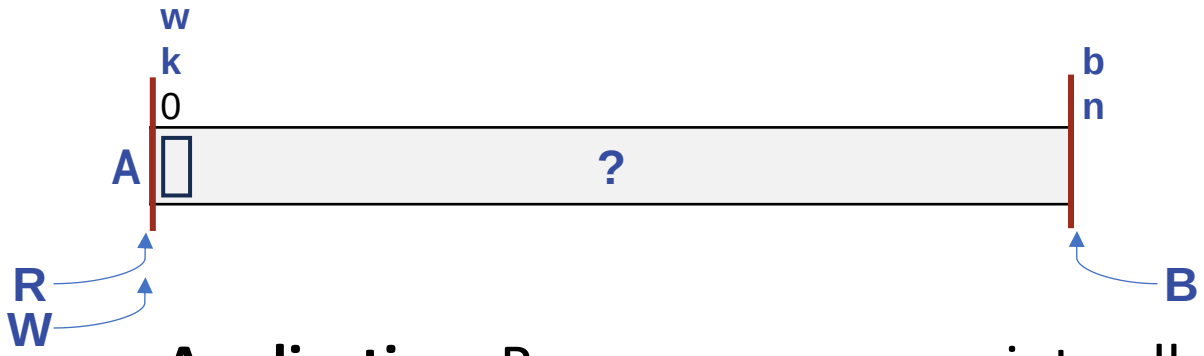
   while ( k!=b )
      if ( A[k]==B ) {
         /* Swap A[b-1] and A[k]. */
         b--;
         }
      else if ( A[k]==R ) {
         /* Swap A[w] and A[k]. */
         w++; k++;
         }
      else /* A[k]==W */
         k++;
```



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   and blue), rearrange A into all red, then white, then blue. */
   /* INVARIANT: A[0..w-1] red, A[w..k-1] white, A[b..n-1] blue, for 0≤w≤k≤b≤n. */

   int k = 0; int w = 0; int b = n;
   while ( k!=b )
      if ( A[k]==B ) {
         /* Swap A[b-1] and A[k]. */
         b--;
         }
      else if ( A[k]==R ) {
         /* Swap A[w] and A[k]. */
         w++; k++;
         }
      else /* A[k]==W */
         k++;
```

Dutch National Flag

Application: Rearrange an array into all red, then all white, then all blue.

```
/* Given array A[0..n-1] consisting of only three values (red, white,
   and blue), rearrange A into all red, then white, then blue. */
   /* INVARIANT: A[0..w-1] red, A[w..k-1] white, A[b..n-1] blue, for 0≤w≤k≤b≤n. */

   int k = 0; int w = 0; int b = n;
   while ( k!=b )
      if ( A[k]==B ) {
         /* Swap A[b-1] and A[k]. */
         b--;
         }
      else if ( A[k]==R ) {
         /* Swap A[w] and A[k]. */
         w++; k++;
         }
      else /* A[k]==W */
         k++;
```

Deferring low-level details has promoted thinking at a higher level of abstraction, but don't forget to elaborate "all the way down".

Application: Rearrange an array into all red, then all white, then all blue.

```
/* Given array A[0..n-1] consisting of only three values (red, white,
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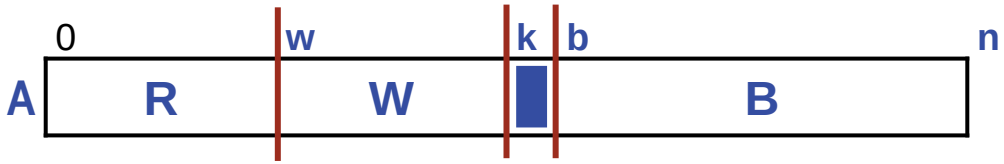
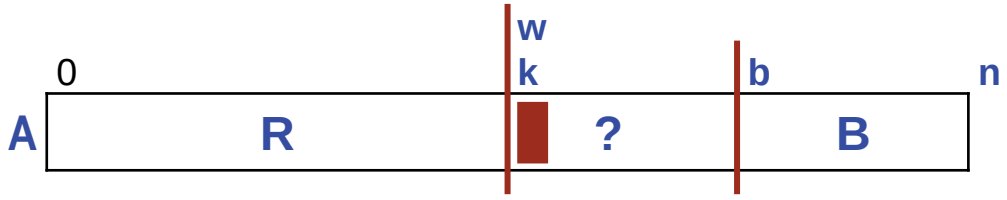
   int k = 0; int w = 0; int b = n;
   while ( k!=b )
      if ( A[k]==B ) {
         /* Swap A[b-1] and A[k]. */
            int temp = A[b-1]; A[b-1] = A[k]; A[k] = temp;
         b--;
         }
      else if ( A[k]==R ) {
         /* Swap A[w] and A[k]. */
            int temp = A[k]; A[k] = A[w]; A[w] = temp;
         w++; k++;
         }
      else /* A[k]==W */
         k++;
```





Boundary Conditions:

What about potential boundary conditions that might have needed special attention?

You can systematically review the code for issues, but will discover that all are nicely treated by the general case.

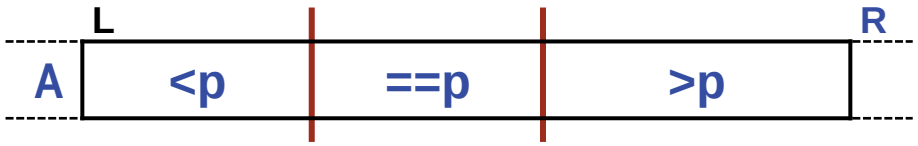
In two situations, `A[k]` gets swapped with itself, but this only happens when its value is already acceptable where it is:

- When the last remaining element in the ? region is blue.
- When `A[k]` is red, and the white region is empty.

All is well.

Performance:

- Constant work per iteration.
- Variant reduced by 1 on each iteration.
- Thus, running time linear in n.

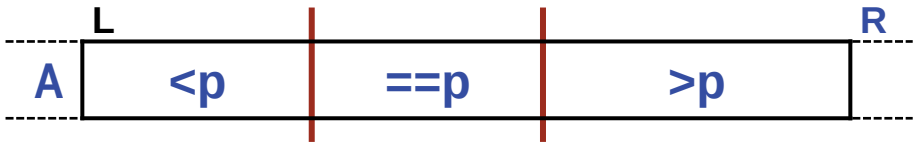


New Application: Rearrange (a segment of) an array into <p, ==p, and >p sections.

```
/* Rearrange A[L..R-1] into all <p, then all ==p, then all >p. */
static void Partition( int A[], int L, int R, int p ) {

   _________
   } /* Partition */
```

☞ **A header-comment says exactly what a method must accomplish, not how it does so.**

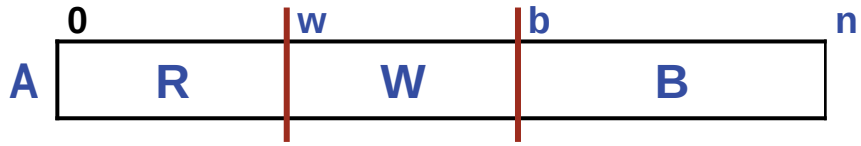


Application: Rearrange (a segment of) an array into <p, ==p, and >p sections.

```
/* Rearrange A[L..R-1] into all <p, then all ==p, then all >p. */
static void Partition( int A[], int L, int R, int p ) {
  ⟨body of Dutch National Flag problem⟩
  } /* Partition */
```

We are simply going to replace the *colors* R, W, and B, by the *properties* "<p", "==p", and ">p".

☞ **Don't type if you can avoid it; clone. Cut and paste, then adapt.**

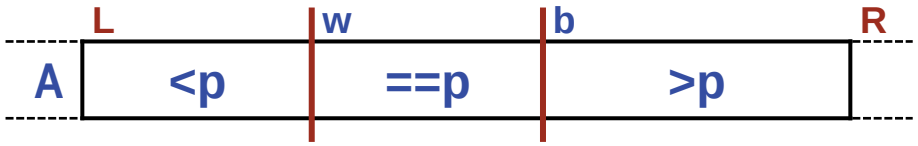


Application: Rearrange (a segment of) an array into <p, ==p, and >p sections.

Dutch National Flag problem

```
/* Rearrange A[L..R-1] into all <p, then all ==p, then all >p. */
static void Partition( int A[], int L, int R, int p ) {
   /* INVARIANT: A[0..w-1] red, A[w..k-1] white, A[b..n-1] blue, for 0≤w≤k≤b≤n. */

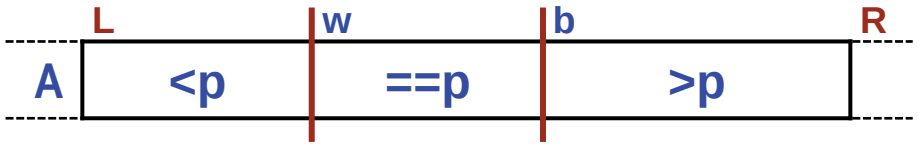
   int k = 0; int w = 0; int b = n;
   while ( k!=b )
      if ( A[k]==B ) {
         /* Swap A[b-1] and A[k]. */
            int temp = A[b-1]; A[b-1] = A[k]; A[k] = temp;
         b--;
         }
      else if ( A[k]==R ) {
         /* Swap A[w] and A[k]. */
            int temp = A[k]; A[k] = A[w]; A[w] = temp;
         w++; k++;
         }
      else /* A[k]==W */
         k++;
   } /* Partition */
```



Application: Rearrange (a segment of) an array into <p, ==p, and >p sections.

```
/* Rearrange A[L..R-1] into all <p, then all ==p, then all >p. */
static void Partition( int A[], int L, int R, int p ) {
   /* INVARIANT: A[0..w-1] is <p, A[w..k-1] is ==p, A[b..n-1] is >p, for 0≤w≤k≤b≤n. */

   int k = L; int w = L; int b = R;
   while ( k!=b )
      if ( A[k]>p ) {
         /* Swap A[b-1] and A[k]. */
            int temp = A[b-1]; A[b-1] = A[k]; A[k] = temp;
         b--;
         }
      else if ( A[k]<p ) {
         /* Swap A[w] and A[k]. */
            int temp = A[k]; A[k] = A[w]; A[w] = temp;
         w++; k++;
         }
      else /* A[k]==p */
         k++;
   } /* Partition */
```



Application: Rearrange (a segment of) an array into <p, ==p, and >p sections.

What value of p would tend to create "<p" and ">p" regions of near equal size?

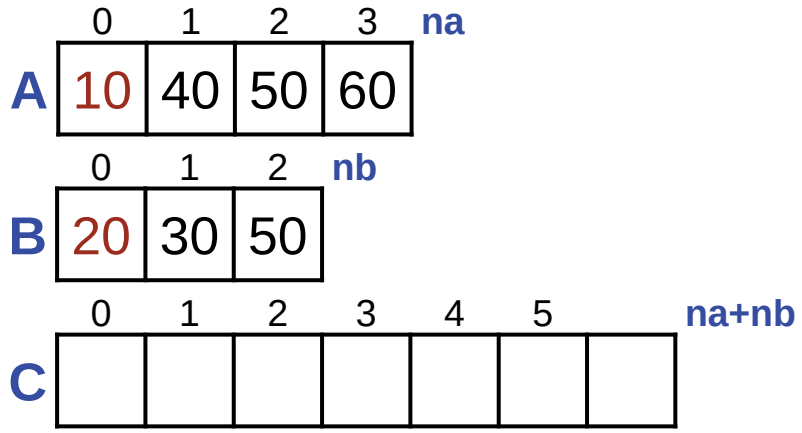
If `A[L..R-1]` are in random order, any of those values is equally good for `p`.

Choosing a p of `(A[L]+A[R-1])/2` guards against poor performance when `A[L..R-1]` is already ordered.

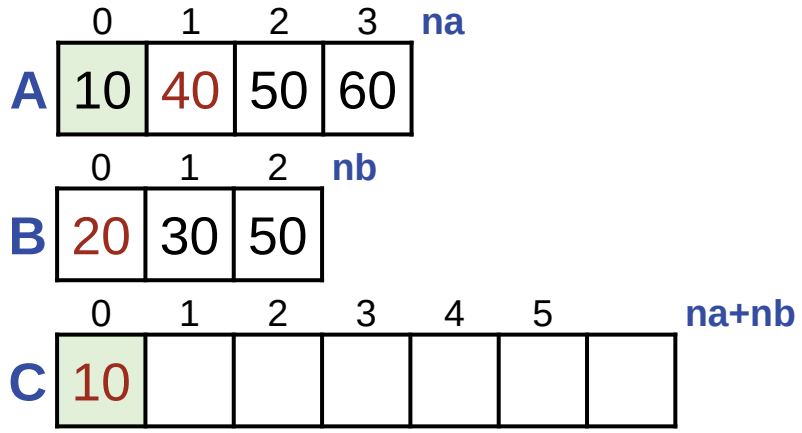
This is central to a Divide and Conquer approach to sorting called QuickSort.

New Application: Collate ordered arrays A and B into array C.

```
/* Given ordered arrays A and B of lengths na and nb, create ordered
   array C of length na+nb consisting of those values. */
```



Application: Collate ordered arrays A and B into array C.



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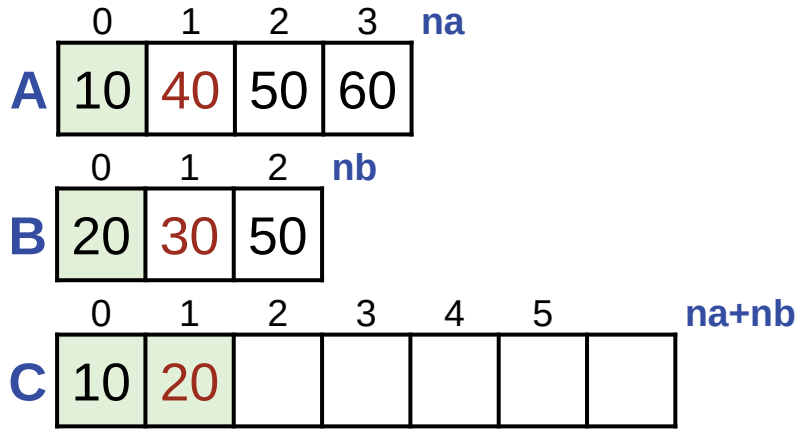

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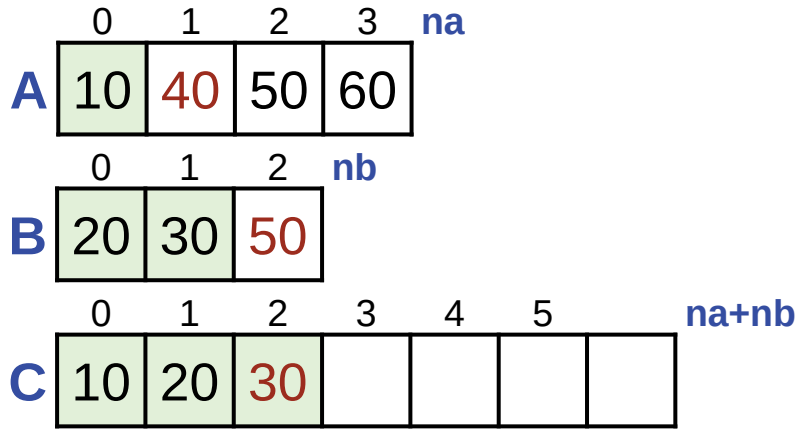

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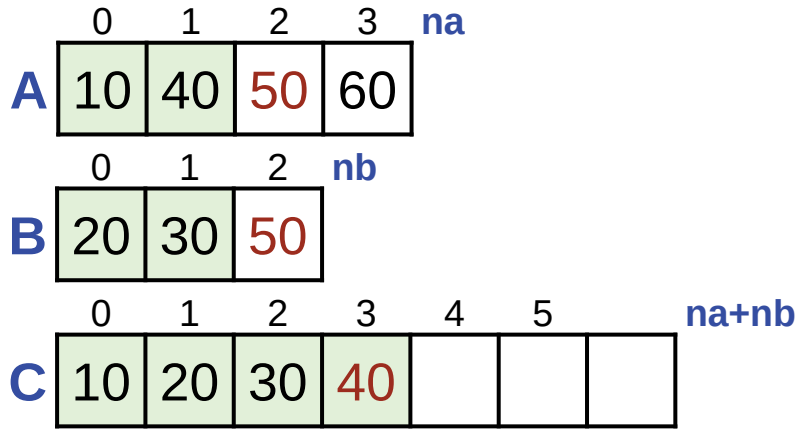

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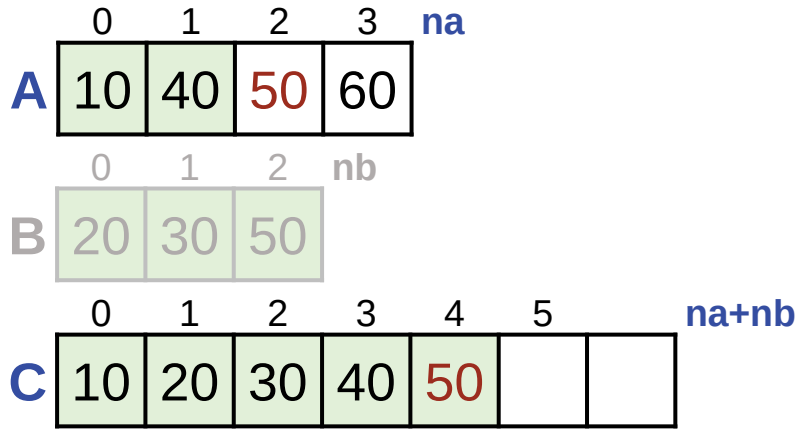

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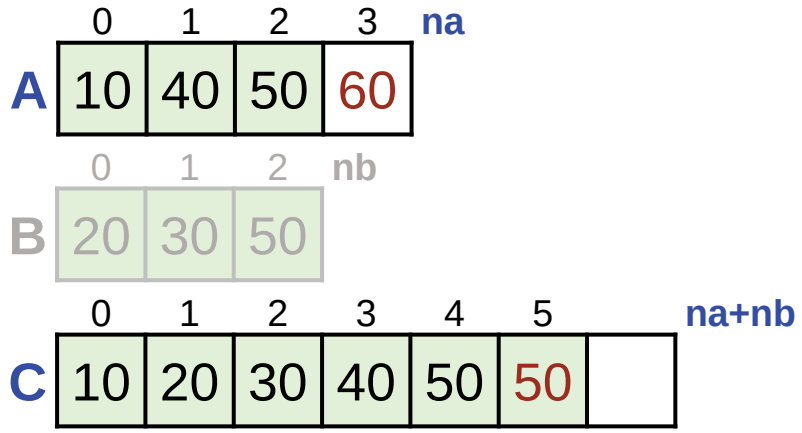

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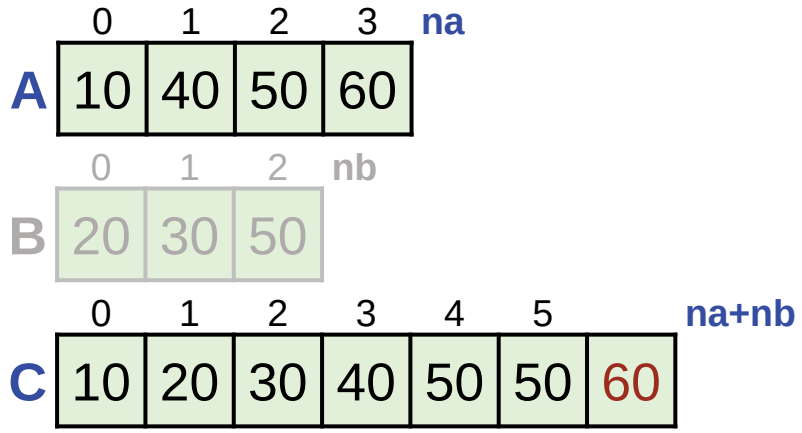
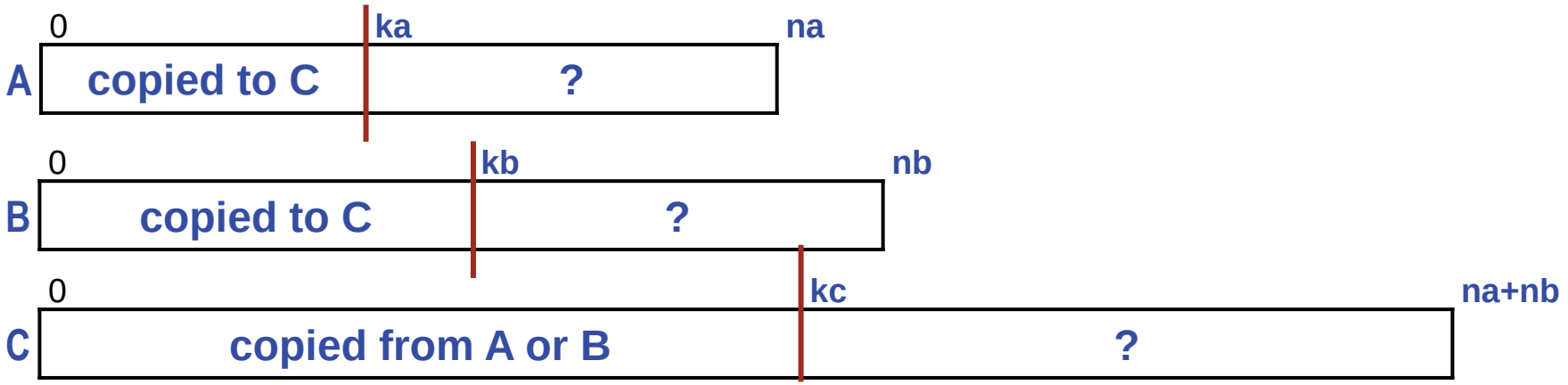


Application: Collate ordered arrays A and B into array C.

Collation is central to a Divide and Conquer approach to sorting called MergeSort.



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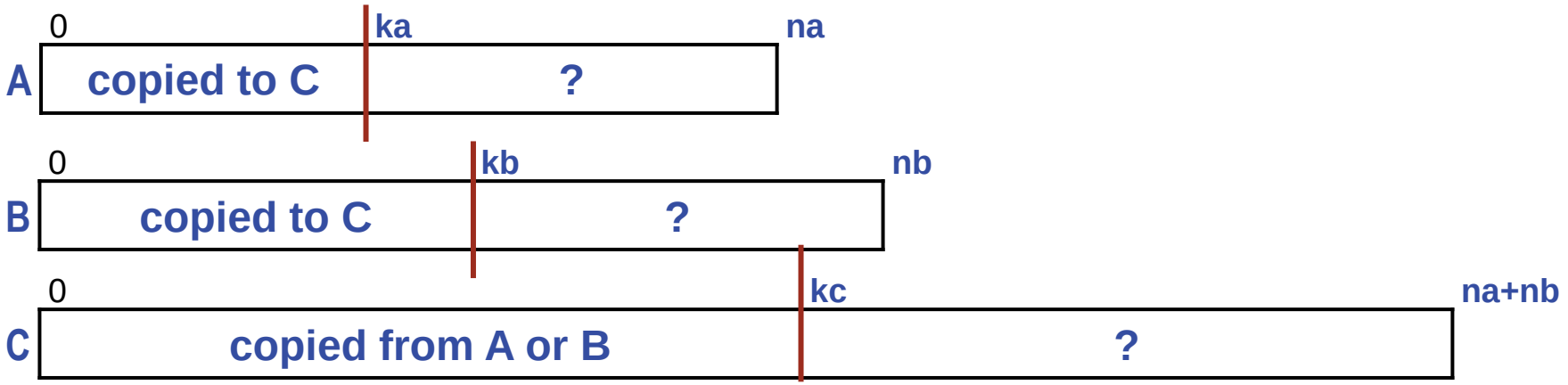

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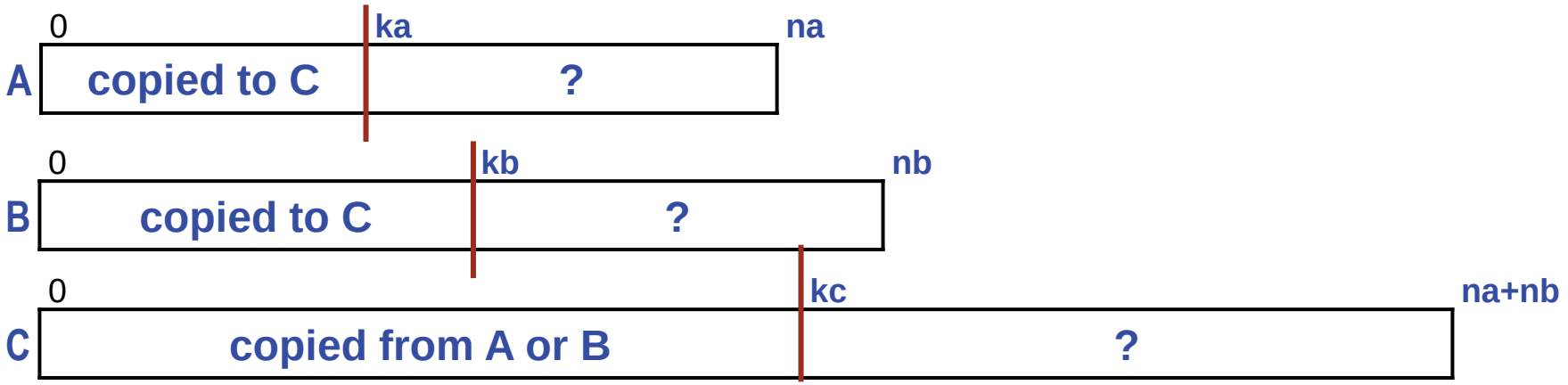

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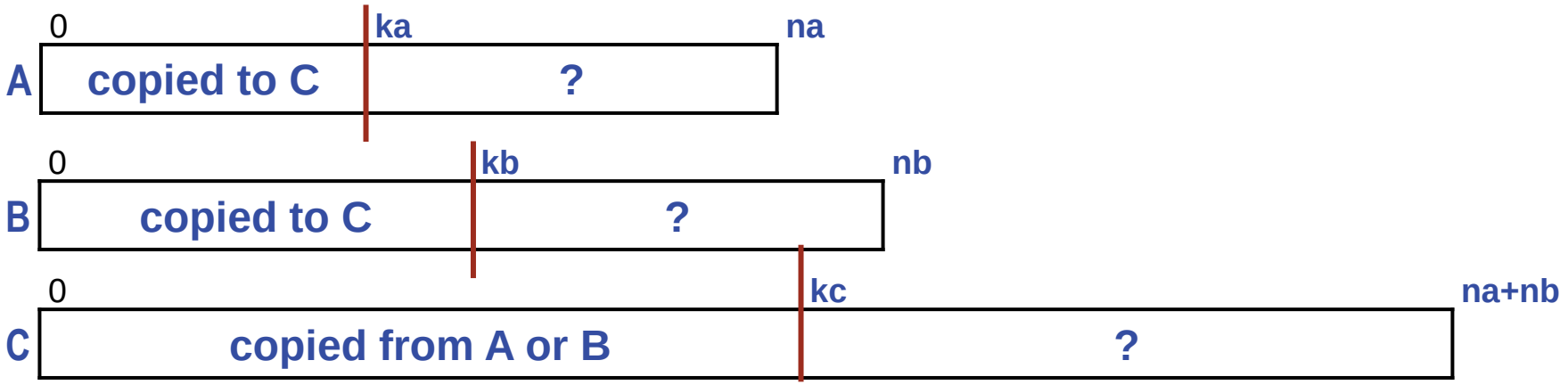

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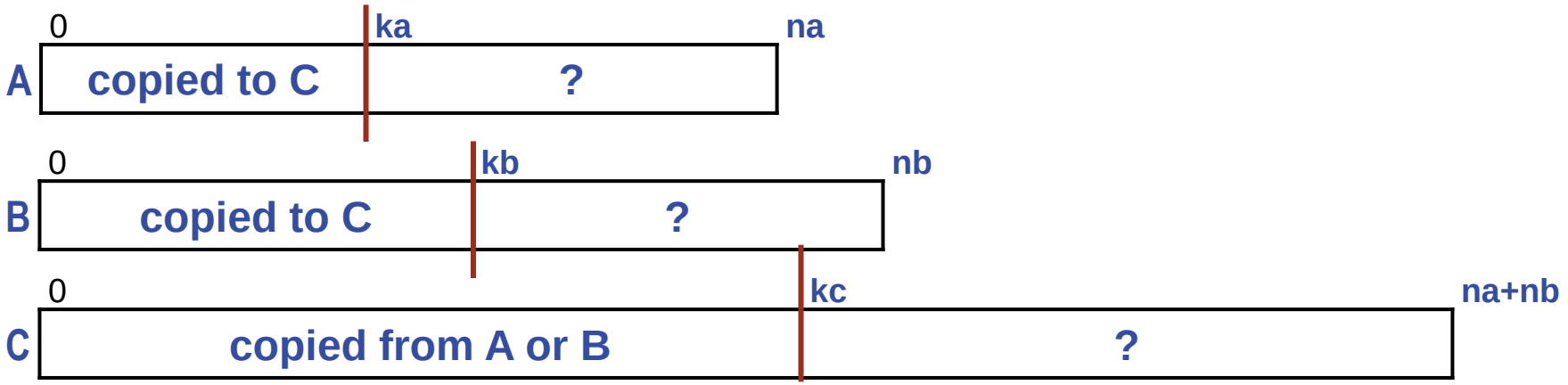

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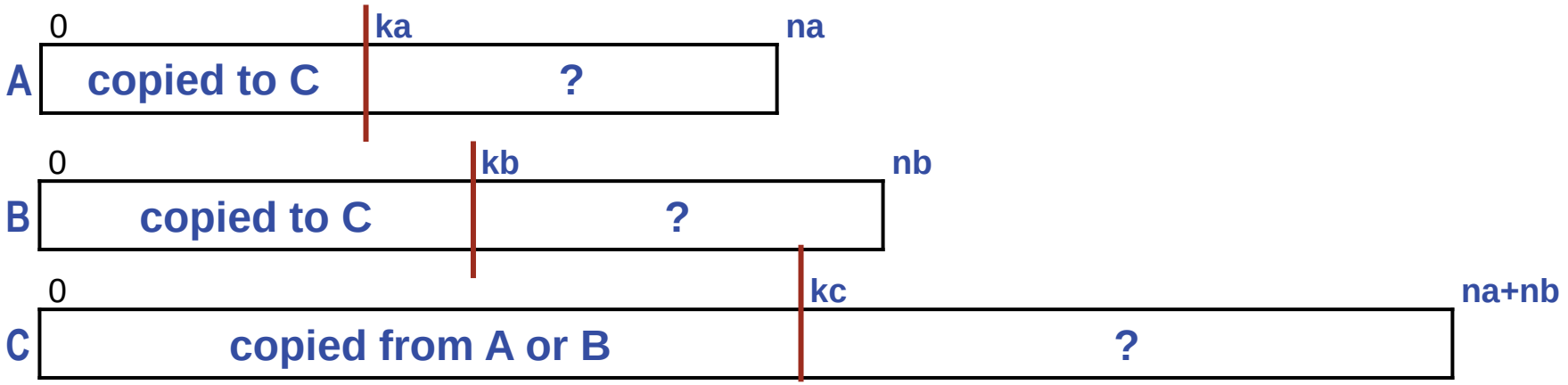

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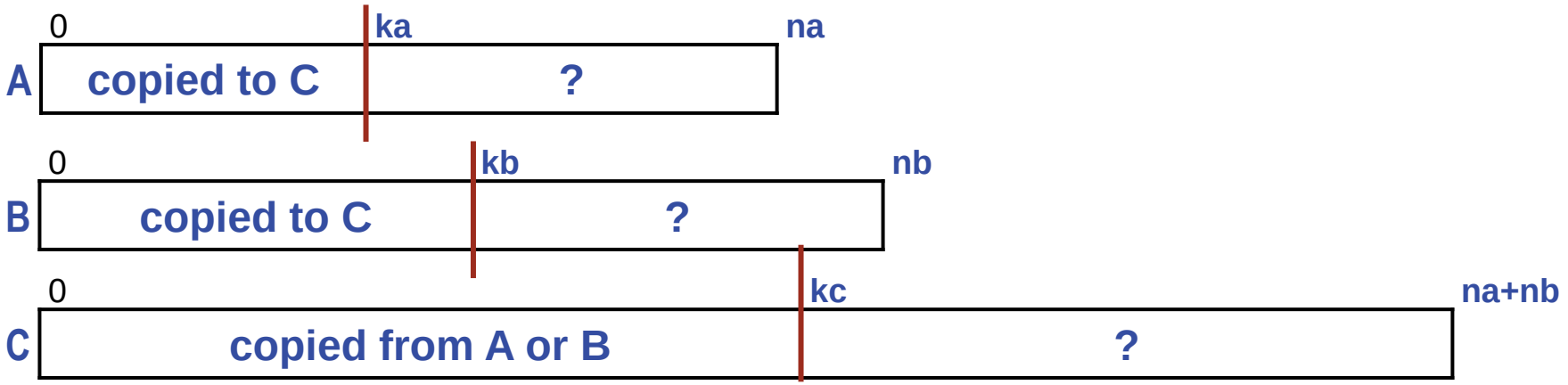

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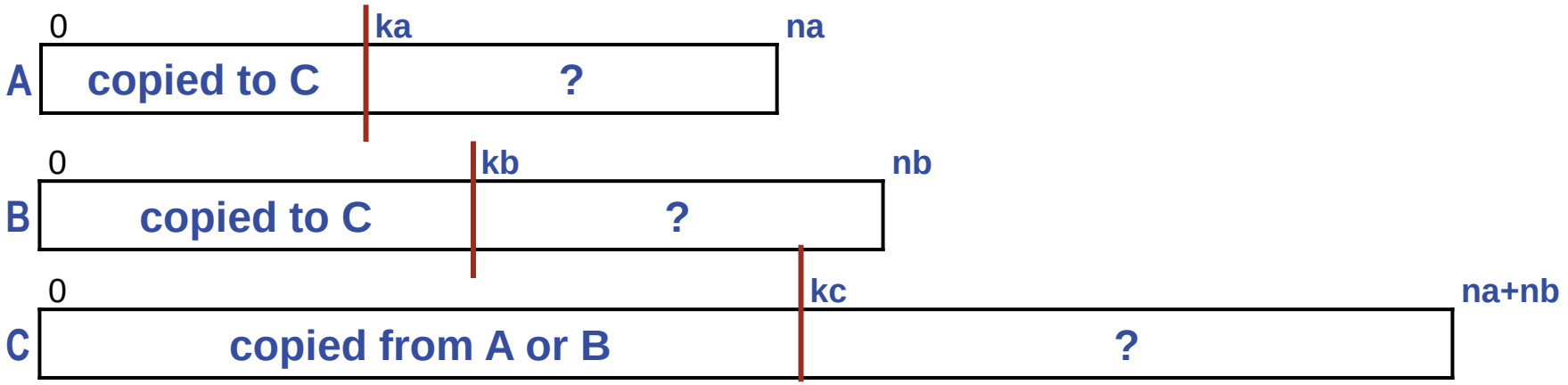

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   /* Copy remaining values into C from the unexhausted array, A or B. */
      while ( ka<na ) { C[kc] = A[ka]; ka++; kc++; }
      while ( kb<nb ) { C[kc] = B[kb]; kb++; kc++; }
```

Summary:

A number of useful one-dimensional array rearrangements were presented, some as methods, and some as code fragments:

- `Reverse(…)`
- `LeftShiftK(…)`
- `LeftRotateOne(…)`
- Left-Rotate-k Four separate implementations were developed and compared.
- Dutch National Flag The basis of Partitioning, and an illustration of invariant-driven programming.
- `Partition(…)` The basis for QuickSelect (Chapter 10), and QuickSort (Chapter 11), and an introduction to an algorithm with good *average-case* performance, and not such good *worst-case* performance.
- Collation The basis for MergeSort (Chapter 11).