Principled Programming

Introduction to Coding in Any Imperative Language

Tim Teitelbaum

Emeritus Professor

Department of Computer Science

Cornell University

Enumeration Patterns

Introduction

To *enumerate* is to list off, one by one.

We consider:

- Counting
- 1-D Indeterminate Enumeration
- 1-D Determinate Enumeration
- 2-D Enumerations

and these applications:

- Sieve of Eratosthenes
- Ramanujan Cubes
- Enumerations of Rational Numbers
- Magic Squares

Counting:

```
int k = 1;
while ( true ) k++;
```

1-origin children

```
int k = 0;
while ( true ) k++;
```

0-origin older children

```
int k = start;
while ( true ) k++;
```

start-origin sophisticated children

Counting:

Linguistic Confusions

		First value enumerated	Number of increments
`int k = 1;` `while ( true ) k++;`	1-origin	1	k-1
`int k = 0;` `while ( true ) k++;`	0-origin	0	k
`int k = start;` `while ( true ) k++;`	*start*-origin	*start*	k-*start*

Off-by-one errors, and their ilk

Number of integers in a range from *first* to *last*, inclusive	*last*-*first*+1
Index of *last* integer in a range of *N* integers starting at 0	*N*-1

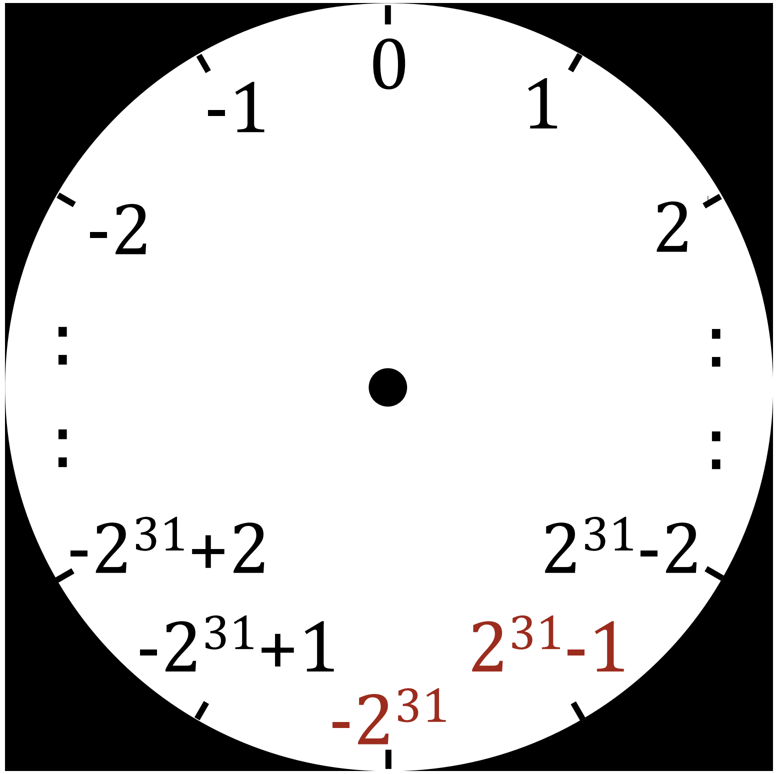
Children learn the concept of infinity from counting. Indeed, these loops run forever, but not because there is no maximum **`int`.** Rather, because after $2^{31}-1$, the next **`int`** is -2^{31}. This is called *arithmetic overflow*.

Counting:

```
int k = 1;
while ( true ) k++;
```

```
int k = 0;
while ( true ) k++;
```

```
int k = start;
while ( true ) k++;
```



From there, counting proceeds “up” to -1, and then around again.

Children learn the concept of infinity from counting. Indeed, these loops run forever, but not because there is no maximum **int.** Rather, because after 2^{31}-1, the next **int** is -2^{31}. This is called *arithmetic overflow.*

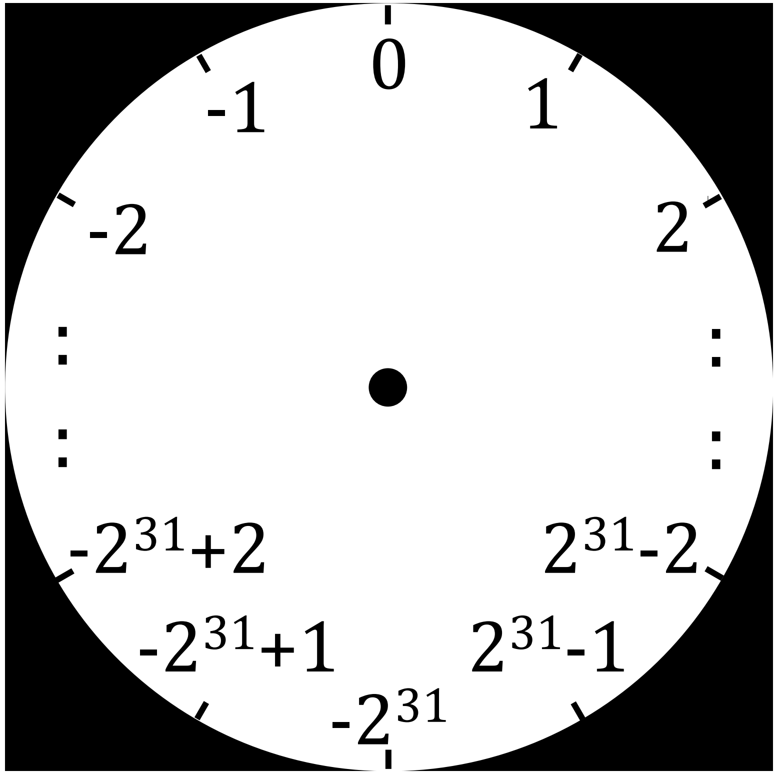
Counting:

```
int k = 1;
while ( true ) k++;
```

```
int k = 0;
while ( true ) k++;
```

```
int k = start;
while ( true ) k++;
```

```
int k = start;
while ( condition ) k++;
```



From there, counting proceeds “up” to -1, and then around again unless *condition* becomes **false** first.

1-D Indeterminate Enumeration:

```
/* Enumerate from start until !condition. */
   int k = start;
   while ( condition ) k++;
```

1-D Indeterminate Enumeration:

```
/* Enumerate from start until !condition, but no further than maximum. */
   int k = start;
   while ( k<=maximum && condition ) k++;

if ( k>maximum ) /* condition was true for all k in [start..maximum]. */
else /* k is smallest in [start..maximum] for which condition is false. */
```

1-D Determinate Enumeration:

```
/* Do whatever n times. */
   int k = 0;
   while ( k<n ) {
      /* whatever */
      k++;
      }
```

or

```
/* Do whatever n times. */
   for (int k=0; k<n; k++)
      /* whatever */
```

1-D Determinate Enumeration

1-D Determinate Enumeration: Don’t terminate a determinate enumeration prematurely.

```
/* Do whatever n times. */
   for (int k=0; k<n; k++) {
      /* whatever */
      if ( condition ) k = n;   // Don’t do this.
      }
```

Rather, do this:

```
k = 0;
while ( k<n && !condition ) {
   /* whatever */
   k++;
   }
```

N.B. The two versions are not exactly equivalent.

Sieve of Eratosthenes

Application of 1-D Determinate Enumeration: Print all primes up to n.

2 3 4 5 6 7 8 9 10 11 12 13 14 15

Consider each integer from 2 through n.

Sieve of Eratosthenes

Application of 1-D Determinate Enumeration: Print all primes up to n.

2 3 4 5 6 7 8 9 10 11 12 13 14 15

Consider each integer from 2 through n.

2 3 **4** 5 **6** 7 **8** 9 **10** 11 **12** 13 **14** 15

If it is not marked out, it is prime: Print it, and mark out all its multiples.

Sieve of Eratosthenes

Application of 1-D Determinate Enumeration: Print all primes up to n.

2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	3	4	5	**6**	7	8	**9**	10	11	**12**	13	14	**15**

Consider each integer from 2 through n.

If it is not marked out, it is prime: Print it, and mark out all its multiples.

Sieve of Eratosthenes

Application of 1-D Determinate Enumeration: Print all primes up to n.

2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	3	4	5	6	7	8	9	**10**	11	12	13	14	**15**

Consider each integer from 2 through n.

If it is not marked out, it is prime: Print it, and mark out all its multiples.

Sieve of Eratosthenes

Application of 1-D Determinate Enumeration: Print all primes up to n.

2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	3	4	5	6	7	8	9	10	11	12	13	**14**	15

Consider each integer from 2 through n.

If it is not marked out, it is prime: Print it, and mark out all its multiples.

Sieve of Eratosthenes

Application of 1-D Determinate Enumeration: Print all primes up to n.

2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	3	4	5	6	7	8	9	10	11	12	13	14	15

Consider each integer from 2 through n.

If it is not marked out, it is prime: Print it, and mark out all its multiples.

Sieve of Eratosthenes

Application of 1-D Determinate Enumeration: Print all primes up to n.

Consider each integer from 2 through n.

2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	3	4	5	6	7	8	9	10	11	12	13	14	15

If it is not marked out, it is prime: Print it, and mark out all its multiples.

Application of 1-D Determinate Enumeration: Print all primes up to n.

```
/* Print primes up to n. */
   /* Initialize sieve to all prime. */
   /* Print each prime in sieve, and cross out its multiples. */
```

Application of 1-D Determinate Enumeration: Print all primes up to n.

```
/* Print primes up to n. */
   /* Initialize sieve to all prime. */
      for (int j=2; j<=n; j++) _____________
   /* Print each prime in sieve, and cross out its multiples. */
```

Application of 1-D Determinate Enumeration: Print all primes up to n.

```
/* Print primes up to n. */
   /* Initialize sieve to all prime. */
      for (int j=2; j<=n; j++) _____________
   /* Print each prime in sieve, and cross out its multiples. */
      for (int j=2; j<=n; j++) _____________
```

Sieve of Eratosthenes

Application of 1-D Determinate Enumeration: Print all primes up to n.

```
/* Print primes up to n. */
   /* Initialize sieve to all prime. */
      for (int j=2; j<=n; j++) _____________
   /* Print each prime in sieve, and cross out its multiples. */
      for (int j=2; j<=n; j++)
         if ( ________ ) {
            System.out.println(j);
            for (int k=2*j; k<=n; k=k+j) _____________
            }
```

Sieve of Eratosthenes

Application of 1-D Determinate Enumeration: Print all primes up to n.

```
/* Print primes up to n. */
   boolean prime[] = new boolean[____];  // prime[k] true iff k is prime.
   /* Initialize sieve to all prime. */
      for (int j=2; j<=n; j++) prime[j] = true;
   /* Print each prime in sieve, and cross out its multiples. */
      for (int j=2; j<=n; j++)
         if ( prime[j] ) {
            System.out.println(j);
            for (int k=2*j; k<=n; k=k+j) prime[k] = false;
            }
```

Sieve of Eratosthenes

Application of 1-D Determinate Enumeration: Print all primes up to n.

```
/* Print primes up to n. */
   boolean prime[] = new boolean[n+1];
   /* Initialize sieve to all prime. */
      for (int j=2; j<=n; j++) prime[j] = true;
   /* Print each prime in sieve, and cross out its multiples. */
      for (int j=2; j<=n; j++)
         if ( prime[j] ) {
            System.out.println(j);
            for (int k=2*j; k<=n; k=k+j) prime[k] = false;
            }
```

2-D Enumerations

Row-major order, determinate enumeration

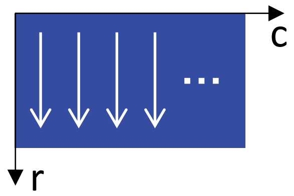
0-origin, e.g., for subscripts

```
/* Enumerate ⟨r,c⟩ in [0..height-1][0..width-1] in row-major order. */
   for (int r=0; r<height; r++)
      for (int c=0; c<width; c++)
         /* whatever */
```

or

1-origin, e.g., for itemization

```
/* Enumerate ⟨r,c⟩ in [1..height][1..width] in row-major order. */
   for (int r=1; r<=height; r++)
      for (int c=1; c<=width; c++)
         /* whatever */
```

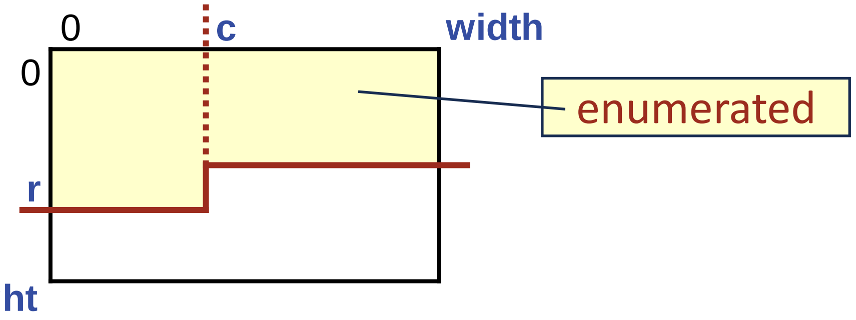


Column-major order, determinate enumeration

0-origin, e.g., for subscripts

```
/* Enumerate ⟨r,c⟩ in [0..height-1][0..width-1] in column-major order. */
   for (int c=0; c<width; c++)
      for (int r=0; r<height; r++)
         /* whatever */
```

INVARIANT:

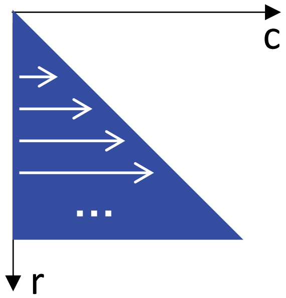


Row-major order, indeterminate enumeration

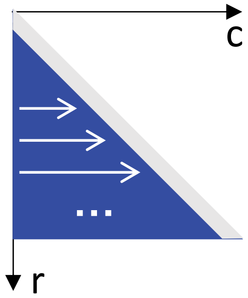
```
/* Enumerate ⟨r,c⟩ in [0..height-1][0..width-1] in row-major order until
   condition, and do whatever for each. */
   int r = 0; int c = 0;
   while ( r<height && !condition ) {
      /* whatever */
      if ( c<width-1 ) c++;  // Not the end of a row; go to next column.
      else { c = 0; r++; }   // The end of a row; go to start of next row.
      }
if ( r==height ) /* fail */ else /* succeed */
```

2-D Enumerations

Triangular order

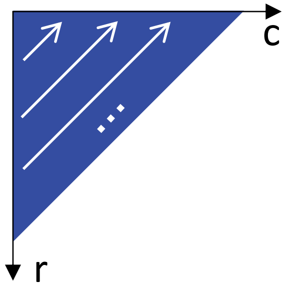



```
/* Enumerate ⟨r,c⟩ in a closed lower-triangular region
   of [0..size-1][0..size-1] in row-major order.*/
   for (int r=0; r<size; r++)
      for (int c=0; c<=r; c++)
         /* whatever */
```




```
/* Enumerate ⟨r,c⟩ in an open lower-triangular region
   of [0..size-1][0..size-1] in row-major order.*/
   for (int r=1; r<size; r++)
      for (int c=0; c<r; c++)
         /* whatever */
```

Think of the enumeration as all ways of choosing two distinct values from [0..size-1].

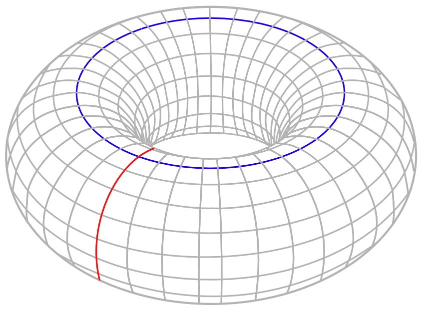


Diagonal order

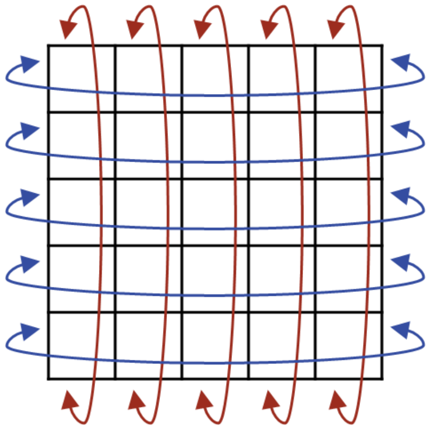
```
/* Unbounded enumeration of ordered ⟨r,c⟩ starting at ⟨0,0⟩ until condition. */
   int d = 0;
   while ( !condition ) {
      int r = d;
      for (int c=0; c<=d; c++) {
         /* whatever */
         r--;
         }
      d++;
      }
```

Think of d as the index of the diagonal.

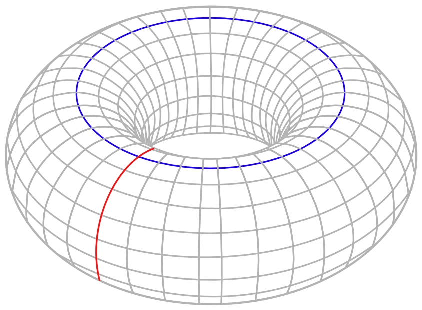
2-D Enumerations



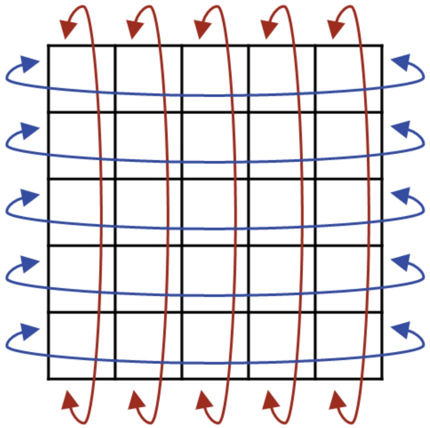
2-D array on a torus

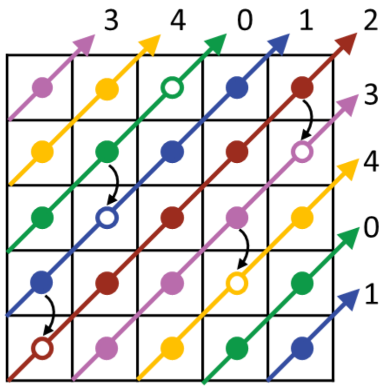


Row and column subscripts wrap around, i.e., after the right-most column comes the left-most column, and after the bottom-most row comes the top-most row.



Toroidal diagonal order






```
/* n-by-n toroidal diagonal-order enumeration in "magical order". */
   int r = 0; int c = n/2;
   for (int d=0; d<n; d++) {
      for (int k=0; k<n; k++) {
         /* whatever */
         r = (r+n-1)%n; c = (c+1)%n;   // up 1 and right 1.
         }
      r = (r+2)%n; c = (c+n-1)%n;      // down 2 and left 1.
      }
```

Application of triangular-order enumeration: We wish to confirm Ramanujan's claim that 1729 is the smallest number that is the sum of two positive cubes in two different ways.

- The integer part of the cube root of 1729 is 12. Thus, we only need to consider the cubes of positive integers that are no larger than 12.
- Let r^3 and c^3 be the two cubes.

Application of triangular-order enumeration:

```
/* Confirm Ramanujan’s claim that 1729 is the smallest number that is the
   sum of two positive cubes in two different ways. */
   /* Record the values of r^3+c^3 that arise for all sets {r,c} of
      distinct positive integers that are no larger than 12. */
   /* Confirm that 1729 is the smallest integer that arose twice. */
```

Application of triangular-order enumeration:

```
/* Confirm Ramanujan's claim that 1729 is the smallest number that is the
   sum of two positive cubes in two different ways. */
   /* Record the values of r^3+c^3 that arise for all sets {r,c} of
      distinct positive integers that are no larger than 12. */
      for (int r=2; r<13; r++)
         for (int c=1; c<r; c++)
            /* Keep track of having seen r^3+c^3. */
   /* Confirm that 1729 is the smallest integer that arose twice. */
```

We complete this code in Chapter 12.

Application of diagonal-order enumeration: We wish to enumerate positive rational numbers.

Start with an enumeration of positive fractions.

1/1	1/2	1/3	1/4	1/5	…
2/1	2/2	2/3	2/4	2/5	…
3/1	3/2	3/3	3/4	3/5	…
4/1	4/2	4/3	4/4	4/5	…
5/1	5/2	5/3	5/4	5/5	…
…	…	…	…	…	…

There are, of course, an infinite number of numerators and denominators, so a row-major-order or column-major-order enumeration won’t do.

Rational Numbers

Application of diagonal-order enumeration: We wish to enumerate positive rational numbers.

Start with an enumeration of positive fractions.

1/1	1/2	1/3	1/4	1/5	…
2/1	2/2	2/3	2/4	2/5	…
3/1	3/2	3/3	3/4	3/5	…
4/1	4/2	4/3	4/4	4/5	…
5/1	5/2	5/3	5/4	5/5	…
…	…	…	…	…	…

There are, of course, an infinite number of numerators and denominators, so a row-major-order or column-major-order enumeration won't do.

A diagonal-order enumeration allows both the numerators and denominators to grow without bound.

Application of diagonal-order enumeration: We wish to enumerate positive rational numbers.

Start with an enumeration of positive fractions.

1/1	1/2	1/3	1/4	1/5	…
2/1	2/2	2/3	2/4	2/5	…
3/1	3/2	3/3	3/4	3/5	…
4/1	4/2	4/3	4/4	4/5	…
5/1	5/2	5/3	5/4	5/5	…
…	…	…	…	…	…

```
/* Output positive fractions. */
   int d = 0;
   while ( true ) {
      int r = d;
      for (int c=0; c<=d; c++) {
         System.out.println( (r+1) + "/" + (c+1) );
         r--;
         }
      d++;
      }
```

Application of diagonal-order enumeration: We wish to enumerate positive rational numbers.

Start with an enumeration of positive fractions.

1/1	1/2	1/3	1/4	1/5	…
2/1	2/2	2/3	2/4	2/5	…
3/1	3/2	3/3	3/4	3/5	…
4/1	4/2	4/3	4/4	4/5	…
5/1	5/2	5/3	5/4	5/5	…
…	…	…	…	…	…

```
/* Output positive fractions. */
   int d = 0;
   while ( true ) {
      int r = d;
      for (int c=0; c<=d; c++) {
         System.out.println( (r+1) + "/" + (c+1) );
         r--;
         }
      d++;
      }
```

However, this lists each rational more than once.

Application of diagonal-order enumeration: We wish to enumerate positive rational numbers.

```
/* Output positive fractions. */
   int d = 0;
   while ( true ) {
      int r = d;
      for (int c=0; c<=d; c++) {
         System.out.println( (r+1) + "/" + (c+1) );
         r--;
         }
      d++;
      }
```

To avoid duplicate listings, we can:

Application of diagonal-order enumeration: We wish to enumerate positive rational numbers.

```
/* Output positive rationals. */
   int d = 0;
   /* set reduced = { }; */
   while ( true ) {
      int r = d;
      for (int c=0; c<=d; c++) {
         /* Let z be the reduced form of the fraction (r+1)/(c+1). */
            int g = gcd(r, c+1);
            /* rational z = ⟨(r+1)/g,(c+1)/g⟩; */
         if ( /* z is not an element of reduced ) {
            System.out.println( (r+1) + "/" + (c+1) );
            /* reduced = reduced ∪ {z}; */
            }
         r--;
         }
      d++;
      }
```

To avoid duplicate listings, we can:

- Maintain the set of reduced fractions already listed.
- Only list a fraction if its reduced form is not in the set.

Application of diagonal-order enumeration: We wish to enumerate positive rational numbers.

```
/* Output positive rationals. */
   int d = 0;
   /* set reduced = { }; */
   while ( true ) {
      int r = d;
      for (int c=0; c<=d; c++) {
         /* Let z be the reduced form of the fraction (r+1)/(c+1). */
            int g = gcd(r, c+1);
            /* rational z = ⟨(r+1)/g,(c+1)/g⟩; */
         if ( /* z is not an element of reduced ) {
            System.out.println( (r+1) + "/" + (c+1) );
            /* reduced = reduced ∪ {z}; */
            }
         r--;
         }
      d++;
      }
```

This introduces two key ideas:

- User-defined types, e.g., `rational`.
- User-defined types that are collections, e.g., `set`.

Application of diagonal-order enumeration: We wish to enumerate positive rational numbers.

```
/* Output positive rationals. */
   int d = 0;
   /* set reduced = { }; */
   while ( true ) {
      int r = d;
      for (int c=0; c<=d; c++) {
         /* Let z be the reduced form of the fraction (r+1)/(c+1). */
            int g = gcd(r, c+1);
            /* rational z = ⟨(r+1)/g,(c+1)/g⟩; */
         if ( /* z is not an element of reduced ) {
            System.out.println( (r+1) + "/" + (c+1) );
            /* reduced = reduced ∪ {z}; */
            }
         r--;
         }
      d++;
      }
```

There are better ways to have proceeded, which we will ignore for pedagogical purposes until Chapter 18.

Application of toroidal diagonal-order enumeration: *n*-by-*n* Magic Squares, for odd *n*.

			15
8	1	6	15
3	5	7	15
4	9	2	15
15	15	15	15

A square grid of numbers is a Magic Square if all rows, columns, and both diagonals sum to the same value.

Application of toroidal diagonal-order enumeration: *n*-by-*n* Magic Squares, for odd *n*.

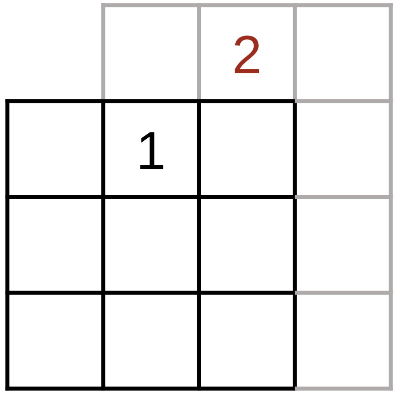
			15
8	1	6	15
3	5	7	15
4	9	2	15
15	15	15	15

	1	

To make an *n*-by-*n* Magic Square, for odd *n*, start with a 1 in the middle of the top row.

Application of toroidal diagonal-order enumeration: *n*-by-*n* Magic Squares, for odd *n*.

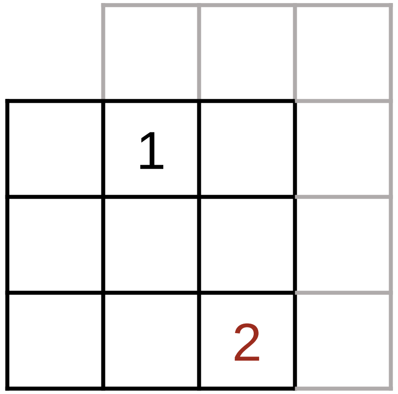
			15
8	1	6	15
3	5	7	15
4	9	2	15
15	15	15	15



To make an *n*-by-*n* Magic Square, for odd *n*, start with a 1 in the middle of the top row, and count up as you proceed diagonally up and to the right (on the surface of a torus).

Application of toroidal diagonal-order enumeration: *n*-by-*n* Magic Squares, for odd *n*.

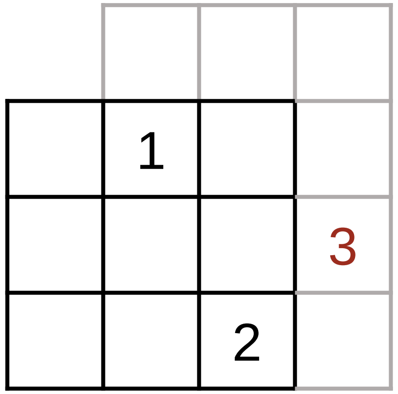
			15
8	1	6	15
3	5	7	15
4	9	2	15
15	15	15	15



To make an *n*-by-*n* Magic Square, for odd *n*, start with a 1 in the middle of the top row, and count up as you proceed diagonally up and to the right (on the surface of a torus).

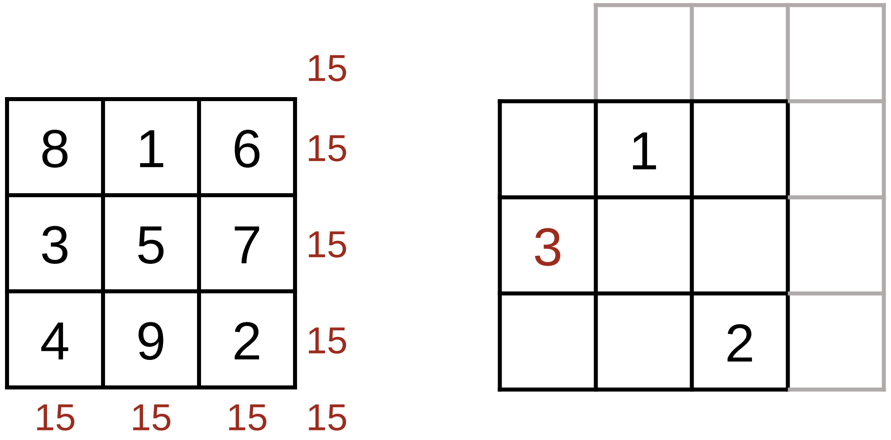
Application of toroidal diagonal-order enumeration: *n*-by-*n* Magic Squares, for odd *n*.

			15
8	1	6	15
3	5	7	15
4	9	2	15
15	15	15	15



To make an *n*-by-*n* Magic Square, for odd *n*, start with a 1 in the middle of the top row, and count up as you proceed diagonally up and to the right (on the surface of a torus).

Application of toroidal diagonal-order enumeration: *n*-by-*n* Magic Squares, for odd *n*.



To make an *n*-by-*n* Magic Square, for odd *n*, start with a 1 in the middle of the top row, and count up as you proceed diagonally up and to the right (on the surface of a torus).

Application of toroidal diagonal-order enumeration: *n*-by-*n* Magic Squares, for odd *n*.

			15
8	1	6	15
3	5	7	15
4	9	2	15
15	15	15	15

	1	
3		
4		2

To make an *n*-by-*n* Magic Square, for odd *n*, start with a 1 in the middle of the top row, and count up as you proceed diagonally up and to the right (on the surface of a torus). When you encounter an already-filled cell, move to the row below (in the same column).

Application of toroidal diagonal-order enumeration: *n*-by-*n* Magic Squares, for odd *n*.

			15
8	1	6	15
3	5	7	15
4	9	2	15
15	15	15	15

	1		
3	5		
4		2	

To make an *n*-by-*n* Magic Square, for odd *n*, start with a 1 in the middle of the top row, and count up as you proceed diagonally up and to the right (on the surface of a torus). When you encounter an already-filled cell, move to the row below (in the same column).

Application of toroidal diagonal-order enumeration: *n*-by-*n* Magic Squares, for odd *n*.

			15
8	1	6	15
3	5	7	15
4	9	2	15
15	15	15	15

	1	6	
3	5		
4		2	

To make an *n*-by-*n* Magic Square, for odd *n*, start with a 1 in the middle of the top row, and count up as you proceed diagonally up and to the right (on the surface of a torus). When you encounter an already-filled cell, move to the row below (in the same column).

Application of toroidal diagonal-order enumeration: *n*-by-*n* Magic Squares, for odd *n*.

			15
8	1	6	15
3	5	7	15
4	9	2	15
15	15	15	15

			7
	1	6	
3	5		
4		2	

To make an *n*-by-*n* Magic Square, for odd *n*, start with a 1 in the middle of the top row, and count up as you proceed diagonally up and to the right (on the surface of a torus). When you encounter an already-filled cell, move to the row below (in the same column).

Application of toroidal diagonal-order enumeration: *n*-by-*n* Magic Squares, for odd *n*.

			15
8	1	6	15
3	5	7	15
4	9	2	15
15	15	15	15

	1	6
3	5	7
4		2

To make an *n*-by-*n* Magic Square, for odd *n*, start with a 1 in the middle of the top row, and count up as you proceed diagonally up and to the right (on the surface of a torus). When you encounter an already-filled cell, move to the row below (in the same column).

Application of toroidal diagonal-order enumeration: *n*-by-*n* Magic Squares, for odd *n*.

			15
8	1	6	15
3	5	7	15
4	9	2	15
15	15	15	15

	1	6	8
3	5	7	
4		2	

To make an *n*-by-*n* Magic Square, for odd *n*, start with a 1 in the middle of the top row, and count up as you proceed diagonally up and to the right (on the surface of a torus). When you encounter an already-filled cell, move to the row below (in the same column).

Application of toroidal diagonal-order enumeration: *n*-by-*n* Magic Squares, for odd *n*.

			15
8	1	6	15
3	5	7	15
4	9	2	15
15	15	15	15

8	1	6	
3	5	7	
4		2	

To make an *n*-by-*n* Magic Square, for odd *n*, start with a 1 in the middle of the top row, and count up as you proceed diagonally up and to the right (on the surface of a torus). When you encounter an already-filled cell, move to the row below (in the same column).

Application of toroidal diagonal-order enumeration: *n*-by-*n* Magic Squares, for odd *n*.

			15
8	1	6	15
3	5	7	15
4	9	2	15
15	15	15	15

	9		
8	1	6	
3	5	7	
4		2	

To make an *n*-by-*n* Magic Square, for odd *n*, start with a 1 in the middle of the top row, and count up as you proceed diagonally up and to the right (on the surface of a torus). When you encounter an already-filled cell, move to the row below (in the same column).

Application of toroidal diagonal-order enumeration: *n*-by-*n* Magic Squares, for odd *n*.

			15
8	1	6	15
3	5	7	15
4	9	2	15
15	15	15	15

8	1	6	
3	5	7	
4	9	2	

To make an *n*-by-*n* Magic Square, for odd *n*, start with a 1 in the middle of the top row, and count up as you proceed diagonally up and to the right (on the surface of a torus). When you encounter an already-filled cell, move to the row below (in the same column).

Application of toroidal diagonal-order enumeration: *n*-by-*n* Magic Squares, for odd *n*.

```
/* Let M be an N-by-N Magic Square, for odd N≥1. */
   int M[][] = new int[N][N]; // Initialized to zeros.
   int r = 0; int c = N/2;
   for (int k=1; k<=N*N; k++) {
      M[r][c] = k;
      /* Advance ⟨r,c⟩ in toroidal diagonal order. */
      }
```

Application of toroidal diagonal-order enumeration: *n*-by-*n* Magic Squares, for odd *n*.

```
/* Let M be an N-by-N Magic Square, for odd N≥1. */
   int M[][] = new int[N][N]; // Initialized to zeros.
   int r = 0; int c = N/2;
   for (int k=1; k<=N*N; k++) {
      M[r][c] = k;
      /* Advance ⟨r,c⟩ in toroidal diagonal order. */
         if ( M[(r+N-1)%N][(c+1)%N]!=0 ) r = (r+1)%N;
         else { r = (r+N-1)%N; c = (c+1)%N; }
      }
```