Principled Programming

Introduction to Coding in Any Imperative Language

Tim Teitelbaum

Emeritus Professor

Department of Computer Science

Cornell University

Sorting

Introduction

	0	1	2	3	4	5	n
A	50	30	10	60	20	40	

(before)

	0	1	2	3	4	5	n
A	10	20	30	40	50	60	

(after)

To *sort* is to rearrange values according to some defined order.

Sorting an array is a fundamental operation, and a way to do so is built into every language.

We study sorting to illustrate these principles:

- Creativity in code development can be inspired by starting with an invariant.
- Different invariants lead to different algorithms, some better than others.
- Algorithms based on Divide and Conquer can have superior performance.
- Algorithms based on everyday experience can have inferior performance.
- Divide-and-Conquer approaches are naturally implemented by recursive procedures.
- Fast algorithms are not necessarily harder to code than slow algorithms.
- Implementations often draw on established code patterns.
- Precise specifications support careful reasoning during implementation.

	0	1	2	3	4	5	n
A	50	30	10	60	20	40	

(before)

	0	1	2	3	4	5	n
A	10	20	30	40	50	60	

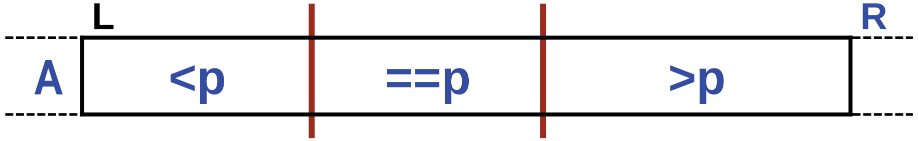
(after)

The specification for sorting an array is:

```
#.Rearrange values of A[0..n-1] into non-decreasing order.
```

We consider four implementations of this specification:

- QuickSort
- Merge Sort
- Selection Sort
- Insertion Sort



Recall that partitioning divides an array segment `A[L..R-1]` into “<p”, “==p”, and “>p” regions.

```
def partition(A: list[int], L: int, R: int, p: int) -> None:
    """
    Given A[L..R-1] and pivot value p, partition(A,L,R,p) rearranges A[L..R-1]
    into all <p, then all ==p, then all >p.
    """

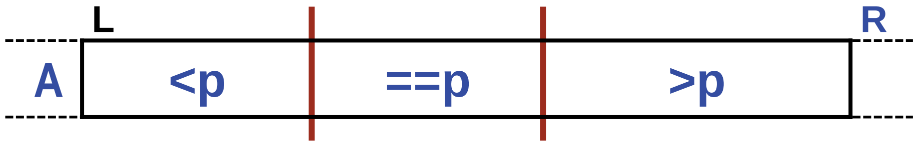
    ⟨body of partition⟩
```

All values in the “<p” region are less than p, which is less than all values in the “>p” region.

Also, on average, appropriate choice of pivot yields “<p” and “>p” regions of near equal size.

This is a basis for a Divide and Conquer algorithm.

☞ **Consider Divide and Conquer when designing an algorithm.**



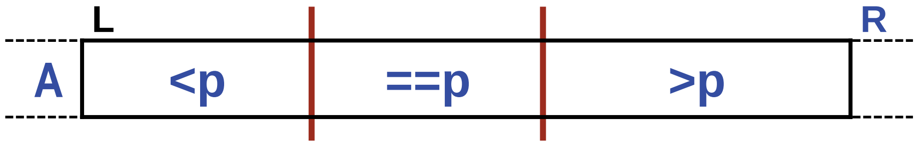
Start with the code for `partition`, and morph it into `quick_sort_aux`:

```
def quick_sort_aux(A: list[int], L: int, R: int) -> None:
    """
    Given A[L..R-1], quick_sort_aux(A,L,R) rearranges A[L..R-1] into
    non-decreasing order.
    """

    p = value-of-pivot
    ⟨body of partition⟩
```

☞ **Don’t type if you can avoid it; clone. Cut and paste, then adapt.**

Change the name and header comment. Move pivot parameter p into the body of the method.



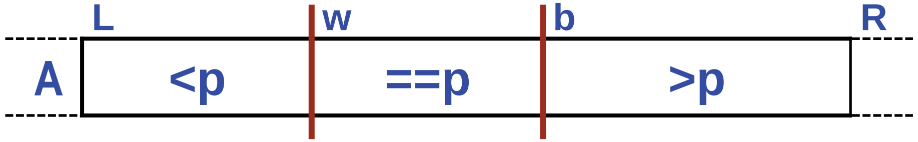
Introduce the base case for regions of size 1 or 0, which perforce are sorted and require no work.

```
def quick_sort_aux(A: list[int], L: int, R: int) -> None:
    """
    Given A[L..R-1], quick_sort_aux(A,L,R) rearranges A[L..R-1] into
    non-decreasing order.
    """

    if R > L:
        p = value-of-pivot
        ⟨body of partition⟩
```

☞ **Don’t type if you can avoid it; clone. Cut and paste, then adapt.**

QuickSort



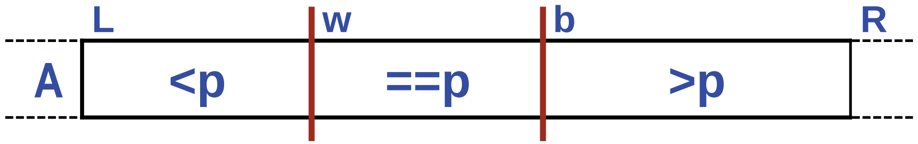
Recursively sort the "<p" and ">p" regions.

```
def quick_sort_aux(A: list[int], L: int, R: int) -> None:
    """
    Given A[L..R-1], quick_sort_aux(A,L,R) rearranges A[L..R-1] into
    non-decreasing order.
    """

    if R > L:
        p = value-of-pivot
        ⟨body of partition⟩
        quick_sort_aux(A, L, w)
        quick_sort_aux(A, b, R)
```

☞ **Consider recursion when designing an algorithm.**

QuickSort

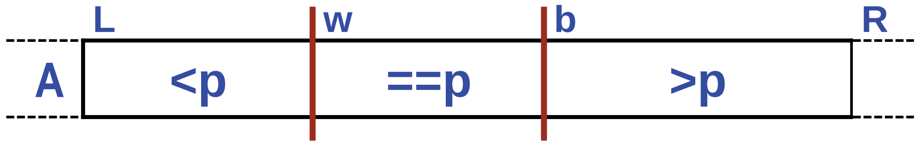


Compute pivot p designed to produce near-equal size “<p” and “>p” regions (on average).

```
def quick_sort_aux(A: list[int], L: int, R: int) -> None:
    """
    Given A[L..R-1], quick_sort_aux(A,L,R) rearranges A[L..R-1] into
    non-decreasing order.
    """

    if R > L:
        p = (A[L] + A[R - 1]) // 2
        ⟨body of partition⟩
        quick_sort_aux(A, L, w)
        quick_sort_aux(A, b, R)
```

QuickSort



Invoke `quick_sort_aux` from the top-level routine `quick_sort`.

```
def quick_sort(A: list[int], n: int) -> None:
    """Rearrange values of A[0..n-1] into non-decreasing order."""
    quick_sort_aux(A, 0, n)
```

Performance: Pivots computed as (A[L] + A[R - 1]) / 2

- Best case. On each iteration, pivot is (serendipitously) the median of `A[L..R-1]`, so region sizes reduced by ½, leading to recursion depth log *n.* At each level of recursion, total partitioning cost is linear in n. Total effort: Proportional to *n* log *n*.
- Worst case. On each iteration, pivot is (serendipitously) the min or max of `A[L..R-1]`, so region sizes reduced by 1, leading to recursion depth *n.* Total effort: $n + (n-1) + (n-2) + \ldots + 1 = n \cdot (n-1)/2$, i.e., quadratic in n.
- Average case, i.e., summed over all permutations of values in `A[0..n-1]`. Total effort: Proportional to *n* log *n.*

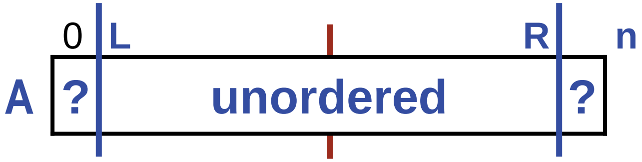
Method `quick_sort` recursively partitions, but region sizes are unpredictable. In contrast, `merge_sort` divides regions into (approximate) halves, quarters, eighths, etc.

```
def merge_sort_aux(A: list[int], L: int, R: int) -> None:
    """Rearrange values of A[L..R] into non-decreasing order."""
```

Note: In analogy with Binary Search, R is changed to the index of the last element of the region rather than one passed the last.

Method `merge_sort` divides (unordered) regions (approximately) in half at each recursion, sorts the halves, and collates those (ordered) halves into an (ordered) whole.

```
def merge_sort_aux(A: list[int], L: int, R: int) -> None.
    """Rearrange values of A[L..R] into non-decreasing order."""
    if R > L:
        m = (L + R) // 2
        merge_sort_aux(A, L, m)
        merge_sort_aux(A, m + 1, R)
        #.Given A[L..m] and A[m+1..R], both already
        #   in non-decreasing order, collate them so
        #   A[L..R] is in non-decreasing order.
```



Method `merge_sort` divides (unordered) regions (approximately) in half at each recursion, sorts the halves, and collates those (ordered) halves into an (ordered) whole.

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def merge_sort_aux(A: list[int], L: int, R: int) -> None.
    """Rearrange values of A[L..R] into non-decreasing order."""
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        m = (L + R) // 2
        merge_sort_aux(A, L, m)
        merge_sort_aux(A, m + 1, R)
        #.Given A[L..m] and A[m+1..R], both already
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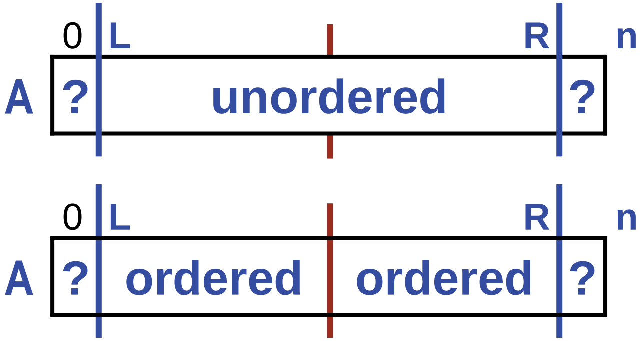
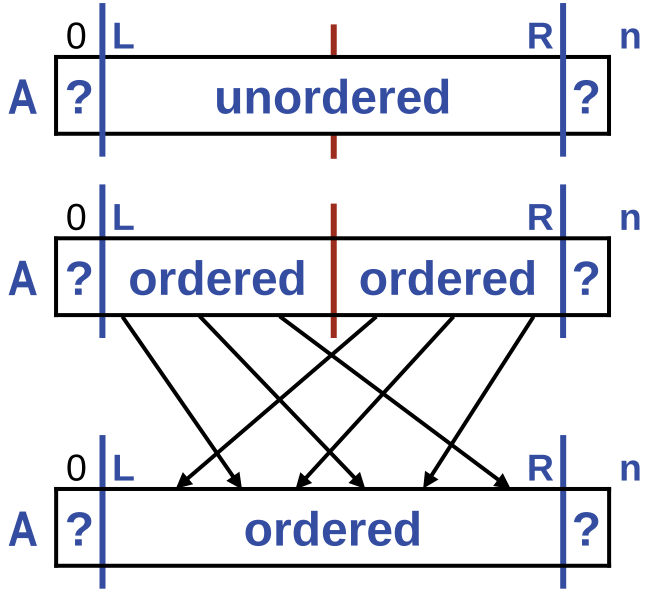
```



Method `merge_sort` divides (unordered) regions (approximately) in half at each recursion, sorts the halves, and collates those (ordered) halves into an (ordered) whole.

```
def merge_sort_aux(A: list[int], L: int, R: int) -> None.
    """Rearrange values of A[L..R] into non-decreasing order."""
    if R > L:
        m = (L + R) // 2
        merge_sort_aux(A, L, m)
        merge_sort_aux(A, m + 1, R)
        #.Given A[L..m] and A[m+1..R], both already
        #   in non-decreasing order, collate them so
        #   A[L..R] is in non-decreasing order.
```

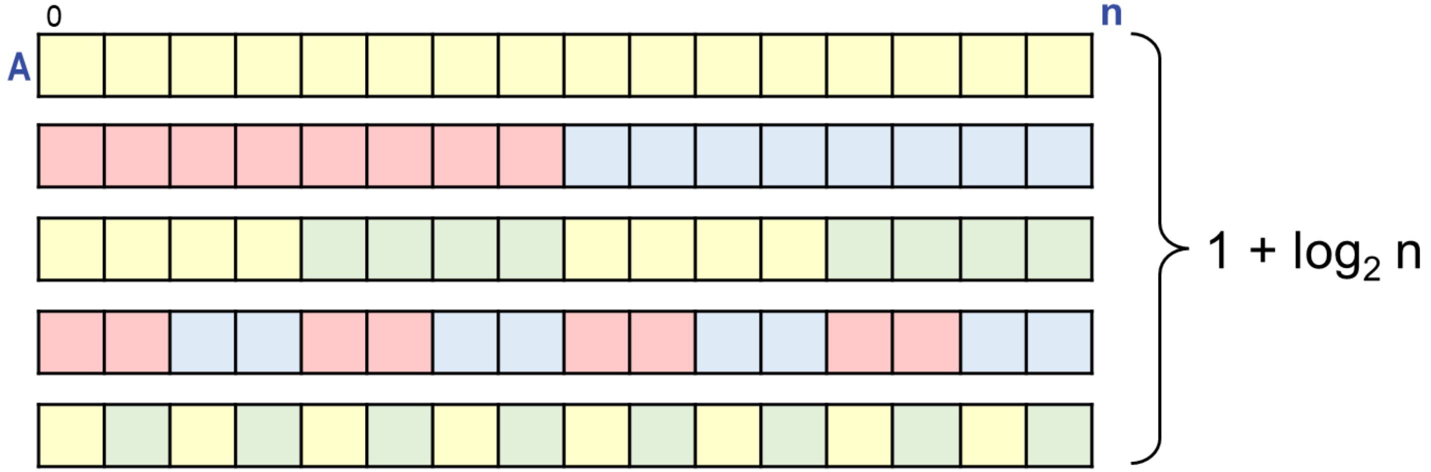


Invoke `merge_sort_aux` from the top-level routine `merge_sort`.

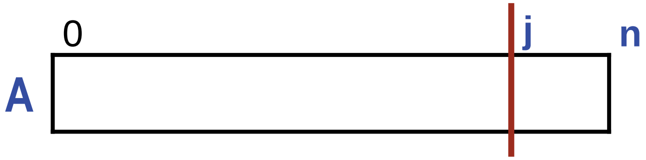
```
def merge_sort(A: list[int], n: int) -> None:
    """Rearrange values of A[0..n-1] into non-decreasing order."""
    merge_sort_aux(A, 0, n - 1)
```

Performance:

- <u>All cases</u>. On each iteration, region sizes reduced by (approximately) ½, leading to recursion depth (approximately) log *n.* At each level of recursion, total collation cost is linear in n. Total effort: Proportional to *n* log *n*.



Positive: Guaranteed *n* log *n* performance. Negative: Not *in situ*.



Selection Sort scans across array A from left to right with index `j`.

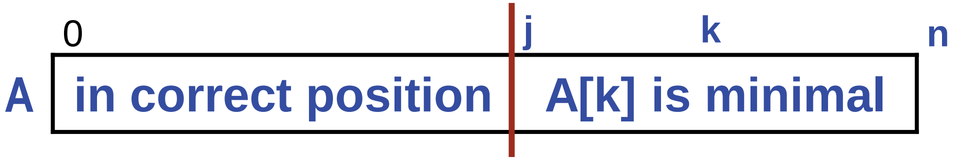
```
# Rearrange values of A[0..n-1] into non-decreasing order.
for j in range(___, ___): _________
```

Selection Sort

INVARIANT: Values in `A[0..j-1]` are in their correct and final positions.

```
# Rearrange values of A[0..n-1] into non-decreasing order.
for j in range(___, ___): __________
```

Selection Sort

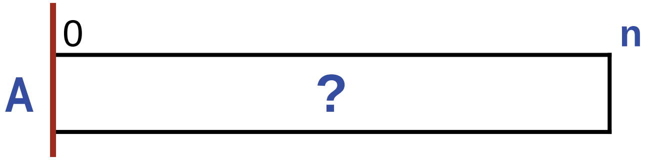


To maintain the **INVARIANT** as `j` is increased by 1, guarantee that `A[j]` is also in its final position.

```
# Rearrange values of A[0..n-1] into non-decreasing order.
for j in range(___, _____):
    #.Let k be s.t. A[k] is a minimal value in A[j..n-1].
    #.Swap A[j] and A[k].
```

If A[0..n-2] are in their correct and final positions, so too is A[n-1].

```
# Rearrange values of A[0..n-1] into non-decreasing order.
for j in range(___, n - 1):
    #.Let k be s.t. A[k] is a minimal value in A[j..n-1]. */
    #.Swap A[j] and A[k].
```



When j==0, the **INVARIANT** that all values in A[0..-1] are in their correct and final positions is trivially true.

```
# Rearrange values of A[0..n-1] into non-decreasing order.
for j in range(0, n - 1):
    #.Let k be s.t. A[k] is a minimal value in A[j..n-1].
    #.Swap A[j] and A[k].
```

Selection Sort

The first step in the loop body is an application of Find Minimal (from Chapter 7).

```
# Rearrange values of A[0..n-1] into non-decreasing order.
for j in range(0, n - 1):
    # Let k be s.t. A[k] is a minimal value in A[j..n-1].
    k = j
    for i in range(j + 1, n):
        if A[i] < A[k]: k = j

    #.Swap A[j] and A[k].
```

Swap is standard.

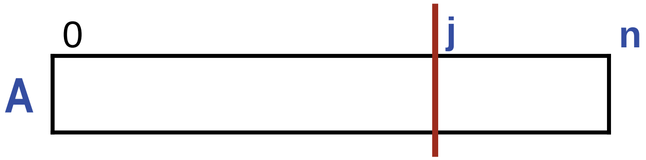
```
# Rearrange values of A[0..n-1] into non-decreasing order.
for j in range(0, n - 1):
    # Let k be s.t. A[k] is a minimal value in A[j..n-1].
    k = j
    for i in range(j + 1, n):
        if A[i] < A[k]: k = j

    # Swap A[j] and A[k]. */
    temp = A[j];  A[j] = A[k];  A[k] = temp
```

Performance: Quadratic in n.

- *All cases*. The sum of the successive efforts to find the minimal value in `A[j..n-1]` is n +(n-1) + (n-2) + … + 2 = n·(n-1)/2 – 1, i.e., proportional to n^2.

Insertion Sort



Insertion Sort scans across array A from left to right with index `j`.

```
# Rearrange values of A[0..n-1] into non-decreasing order.
for j in range(___, ___): _________
```

Insertion Sort

INVARIANT: Values in `A[0..j-1]` are in non-decreasing order.

```
# Rearrange values of A[0..n-1] into non-decreasing order.
for j in range(___, ___): _________
```

Insertion Sort

To maintain the **INVARIANT** as `j` is increased by 1, insert `A[j]` into `A[0..j]` appropriately.

```
# Rearrange values of A[0..n-1] into non-decreasing order.
for j in range(___, ___):
    #.Given A[0..j-1] ordered in non-decreasing order, rearrange values of
    #   A[0..j] so it is ordered.
```

Insertion Sort

The last element of `A[0..n-1]` may have to move, just like the others.

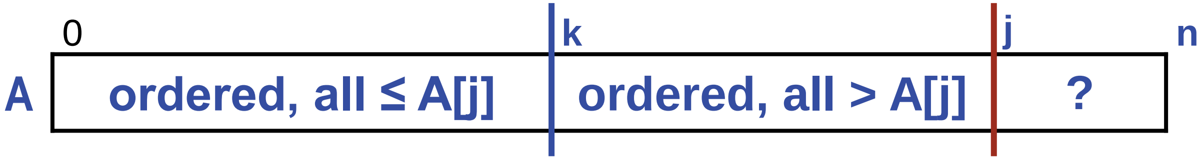
```
# Rearrange values of A[0..n-1] into non-decreasing order.
for j in range(___, n):
    #.Given A[0..j-1] ordered in non-decreasing order, rearrange values of
    #   A[0..j] so it is ordered.
```

Insertion Sort

When `j==1`, the **INVARIANT** that all values in `A[0..0]` is ordered is trivially true.

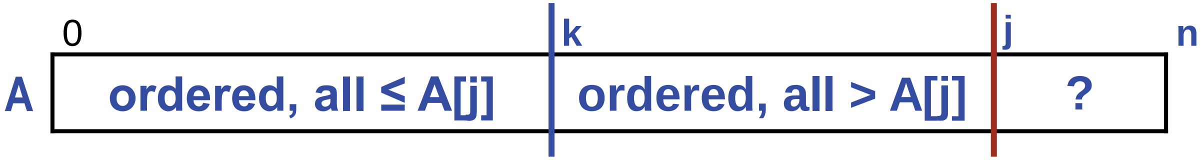
```
# Rearrange values of A[0..n-1] into non-decreasing order.
for j in range(1, n):
    #.Given A[0..j-1] ordered in non-decreasing order, rearrange values of
    #   A[0..j] so it is ordered.
```

Insertion Sort



Right-shift values of `A[0..j-1]` that are larger than `A[j]`. Then insert `A[j]` appropriately.

```
# Rearrange values of A[0..n-1] into non-decreasing order.
for j in range(1, n):
    # Given A[0..j-1] ordered in non-decreasing order, rearrange values of
    #   A[0..j] so it is ordered.
    # ---------------------------------------------------------------------
    temp = A[j]
    #.Shift A[k..j-1] right one place, where k is the largest
    #   integer s.t. A[k-1]≤temp, or 0 if temp is smallest.
    A[k] = temp
```



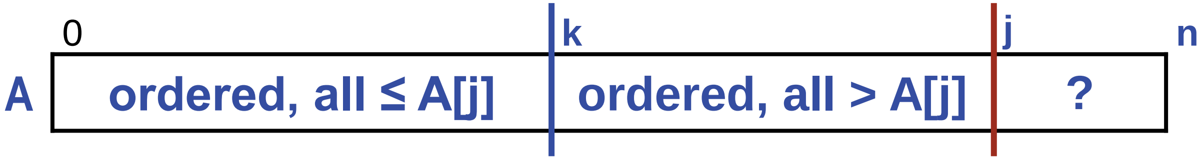
Treat the inner loop as a right-to-left search for rightmost k s.t. `A[k]≤A[j]`.

```
# Rearrange values of A[0..n-1] into non-decreasing order.
for j in range(1, n):
    # Given A[0..j-1] ordered in non-decreasing order, rearrange values of
    #   A[0..j] so it is ordered.
    # ----------------------------------------------------------------------
    temp = A[j]

    # Shift A[k..j-1] right one place, where k is the largest
    #   integer s.t. A[k-1]≤temp, or 0 if temp is smallest.
    k = ____
    while ______:
        A[ ____ ] = A[ ____ ]
        k -= 1

    A[k] = temp
```

Insertion Sort



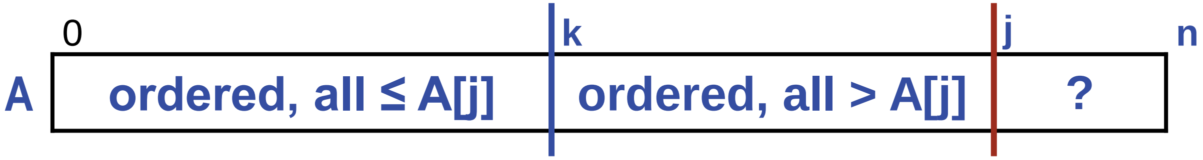
Treat loop as a right-to-left search for rightmost k s.t. `A[k]≤A[j]`.

```
# Rearrange values of A[0..n-1] into non-decreasing order.
for j in range(1, n):
    # Given A[0..j-1] ordered in non-decreasing order, rearrange values of
    #   A[0..j] so it is ordered.
    # ------------------------------------------------------------------
    temp = A[j]

    # Shift A[k..j-1] right one place, where k is the largest
    #   integer s.t. A[k-1]≤temp, or 0 if temp is smallest.
    k = j
    while ______:
        A[ ____ ] = A[ ____ ]
        k -= 1

    A[k] = temp
```

Insertion Sort



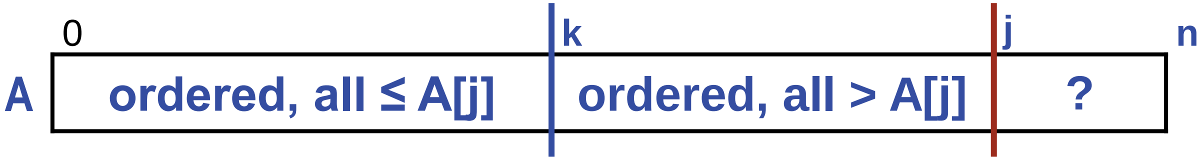
Treat loop as a right-to-left search for rightmost k s.t. `A[k]≤A[j]`.

```
# Rearrange values of A[0..n-1] into non-decreasing order.
for j in range(1, n):
    # Given A[0..j-1] ordered in non-decreasing order, rearrange values of
    #   A[0..j] so it is ordered.
    # ----------------------------------------------------------------------
    temp = A[j]

    # Shift A[k..j-1] right one place, where k is the largest
    #   integer s.t. A[k-1]≤temp, or 0 if temp is smallest.
    k = j
    while A[k - 1] ____ temp:
        A[ ____ ] = A[ ____ ]
        k -= 1

    A[k] = temp
```

Insertion Sort



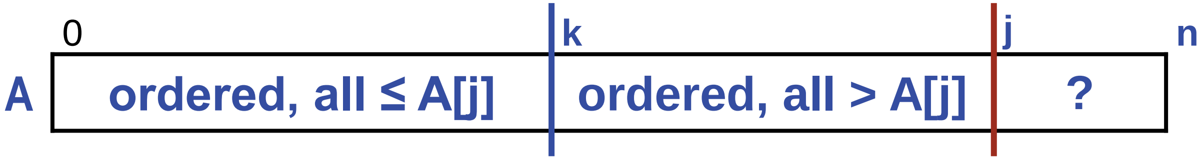
Treat loop as a right-to-left search for rightmost k s.t. `A[k]≤A[j]`.

```
# Rearrange values of A[0..n-1] into non-decreasing order.
for j in range(1, n):
    # Given A[0..j-1] ordered in non-decreasing order, rearrange values of
    #   A[0..j] so it is ordered.
    # ------------------------------------------------------------------
    temp = A[j]

    # Shift A[k..j-1] right one place, where k is the largest
    #   integer s.t. A[k-1]≤temp, or 0 if temp is smallest.
    k = j
    while A[k - 1] > temp:
        A[ ____ ] = A[ ____ ]
        k -= 1

    A[k] = temp
```

Insertion Sort

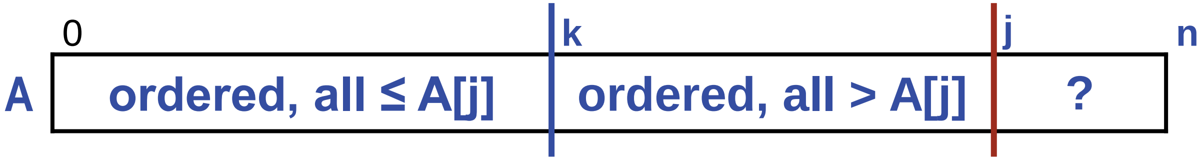


Allow for `A[j]` being minimum.

```
# Rearrange values of A[0..n-1] into non-decreasing order.
for j in range(1, n):
    # Given A[0..j-1] ordered in non-decreasing order, rearrange values of
    #   A[0..j] so it is ordered.
    # ------------------------------------------------------------------
    temp = A[j]

    # Shift A[k..j-1] right one place, where k is the largest
    #   integer s.t. A[k-1]≤temp, or 0 if temp is smallest.
    k = j
    while (k > 0) and (A[k - 1] > temp) :
        A[ ____ ] = A[ ____ ]
        k -= 1

    A[k] = temp
```



Do the shift at the same time as the search. Could end up putting `A[j]` right back where it started.

```
# Rearrange values of A[0..n-1] into non-decreasing order.
for j in range(1, n):
    # Given A[0..j-1] ordered in non-decreasing order, rearrange values of
    #   A[0..j] so it is ordered.
    # ------------------------------------------------------------------
    temp = A[j]

    # Shift A[k..j-1] right one place, where k is the largest
    #   integer s.t. A[k-1]≤temp, or 0 if temp is smallest.
    k = j
    while (k > 0) and (A[k - 1] > temp):
        A[k] = A[k - 1]
        k -= 1

    A[k] = temp
```

Performance: Quadratic in n.

- *<u>Worst case</u>*. Array starts out in non-increasing order. The sum of the successive shifts is $1 + 2 + \ldots + (n\text{-}2) + (n\text{-}1) = n \cdot (n\text{-}1)/2$, i.e., proportional to n^2.
- <u>Best case</u>. Array starts out already ordered. Linear in *n*.