Principled Programming

Introduction to Coding in Any Imperative Language

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Median

	0	1	2	3	4	n
A	10	20	**30**	40	50	

	0	1	2	3	4	5	n
A	10	20	30	**40**	50	60	

The *median* of an ordered array of *n* values is the middle value. If *n* is odd, this is `A[`*n*`//2]`; if *n* is even, we also opt for `A[`*n*`//2]` rather than averaging the middle two values.

(Recall that `//` is integer division, where the fractional part is truncated. Thus, for example, 5//2 is 2, and 6//2 is 3.)

	0	1	2	3	4	n
A	50	**30**	10	40	20	

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(Recall that `//` is integer division, where the fractional part is truncated. Thus, for example, 5//2 is 2, and 6//2 is 3.)

But what if the array is not ordered. How would you find the median then?

You could sort the array and select `A[n//2]`. But sorting requires *n* log *n* operations.

Is it possible to do better? Try it. You will find that everyday experience is no help.

	0	1	2	3	4	n
A	50	**30**	10	40	20	

	0	1	2	3	4	5	n
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The *median* of an ordered array of *n* values is the middle value. If *n* is odd, this is `A[`*n*`//2]`; if *n* is even, we also opt for `A[`*n*`//2]` rather than averaging the middle two values.

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We need principles to follow in such cases.

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	0	1	2	3	4	5	n
A	50	30	10	60	20	**40**	

Three principles that can help are:

- ☞ **Consider generalizing a problem when designing an algorithm.**
- ☞ **Consider Divide and Conquer when designing an algorithm.**
- ☞ **Consider recursion when designing an algorithm.**

We will use them to derive:

- An Average-Case Linear-Time Median Algorithm
- A Worst-Case Linear-Time Median Algorithm

It is astounding that it is possible to find the median of an unordered array of length *n* in linear time, i.e., time proportional to *n*.

The *median* of an ordered array of n values is the middle value. If n is odd, this is `A[`n`//2]`; if n is even, we opt for `A[`n`//2]` rather than averaging the middle two values.

☞ **Consider generalizing a problem when designing an algorithm.**

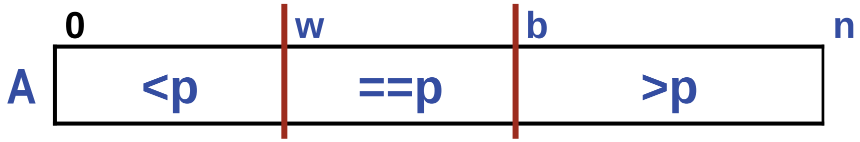
Selection: Given a set of n rank-ordered values, select the `j`th smallest value of the set.

Average-Case Linear-Time Algorithm

Selection: Given a set of *n* rank-ordered values, select the j^{th} smallest value of the set.

☞ **Consider Divide and Conquer when designing an algorithm.**

Recall: Partitioning, based on the Dutch National Flag problem, for some pivot *p*:



$0 \le j < w$. The j^{th} smallest value is the j^{th} smallest value of `A[0..w-1]`
$w \le j < b$. The j^{th} smallest value is the pivot, *p*
$b \le j < n$. The j^{th} smallest value is the $(j-b)^{th}$ smallest value in `A[b..n-1]`

Choose one of the three regions based on a Partition (Divide) and repeat (Conquer).

Start with the code for `Partition`, and morph it into `quick_select`:

```
def partition(A: list[int], L: int, R: int, p: int) -> None:
    """
    Given A[L..R-1] and pivot value p, partition(A,L,R,p) rearranges A[L..R-1]
    into all <p, then all ==p, then all >p.
    """

    ⟨body of partition⟩
```

☞ **Don't type if you can avoid it; clone. Cut and paste, then adapt.**

Start with the code for `Partition`, and morph it into `quick_select`:

```
def quick_select(A: list[int], L: int, R: int, p: int) -> None:
    """
    Given A[0..n-1], and int 0≤j<n, quick_select(A,n,j) returns the j-th smallest
    value in A[0..n-1].
    """
    quick_select
    ⟨body of partition⟩
```

☞ **Don't type if you can avoid it; clone. Cut and paste, then adapt.**

Change the name and docstring.

Start with the code for `Partition`, and morph it into `quick_select`:

```
def quick_select(A: list[int], n: int, j: int) -> None:
    """
    Given A[0..n-1], and int 0≤j<n, quick_select(A,n,j) returns the j-th smallest
    value in A[0..n-1].
    """

    L = 0;  R = n
    p = value-of-pivot
    ⟨body of partition⟩
```

☞ **Don’t type if you can avoid it; clone. Cut and paste, then adapt.**

Move parameters L, R, and p into the body of QuickSelect, and introduce parameters n and j.

Start with the code for `Partition`, and morph it into `quick_select`:

```
def quick_select(A: list[int], n: int, j: int) -> int:
    """
    Given A[0..n-1], and int 0≤j<n, quick_select(A,n,j) returns the j-th smallest
    value in A[0..n-1].
    """

    L = 0;  R = n
    p = value-of-pivot
    ⟨body of partition⟩
    return ____
```

☞ **Don't type if you can avoid it; clone. Cut and paste, then adapt.**

Change return type to **int**, and introduce a **return** statement for the result.

Could consider recursion, but it is not needed because we can just …

```
def quick_select(A: list[int], n: int, j: int) -> int:
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    Given A[0..n-1], and int 0≤j<n, quick_select(A,n,j) returns the j-th smallest
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    L = 0;  R = n
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    ⟨body of partition⟩
    return ____
```

Average-Case Linear-Time Algorithm

INVARIANT

Update L, R, and p iteratively using the **INVARIANT** shown.

```
def quick_select(A: list[int], n: int, j: int) -> int:
    """
    Given A[0..n-1], and int 0≤j<n, quick_select(A,n,j) returns the j-th smallest
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    L = 0;  R = n
    p = value-of-pivot
    ⟨body of partition⟩
    return ____
```
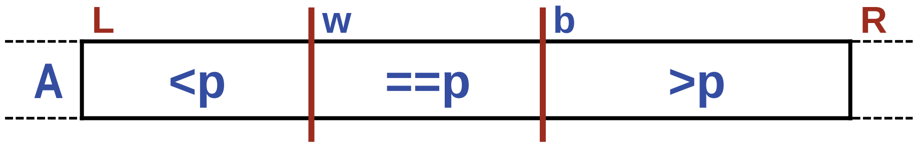

iterant

```
# Initialize.
while not-finished:
    # Compute.
    # Go-on-to-next.
```

iterant



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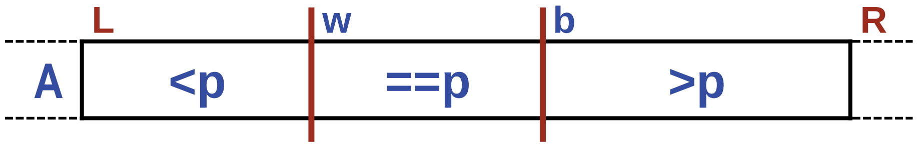

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    L = 0;  R = n
    while not-finished:
        p = value-of-pivot
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        #.Go-on-to “<p” or “>p” region if j-th smallest there; else return p.
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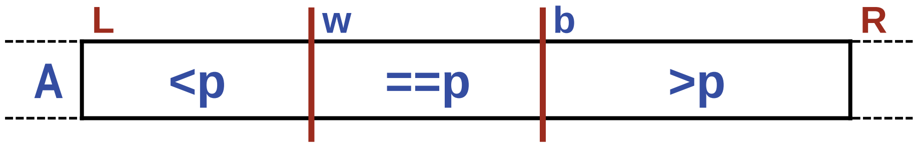

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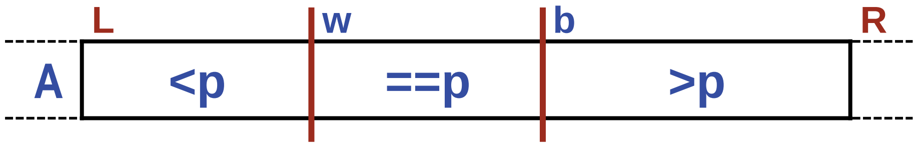
    L = 0;  R = n
    while not-finished:
        p = value-of-pivot
        ⟨body of partition⟩

        # Go-on-to "<p" or ">p" region if j-th smallest there; else return p.
        if j < w: R = w
        elif j < b: return p
        else: L = b
    return ____
```

Average-Case Linear-Time Algorithm



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    value in A[0..n-1].
    """

    L = 0;  R = n
    while (R - L) > 1:
        p = value-of-pivot
        ⟨body of partition⟩

        # Go-on-to “<p” or “>p” region if j-th smallest there; else return p.
        if j < w: R = w
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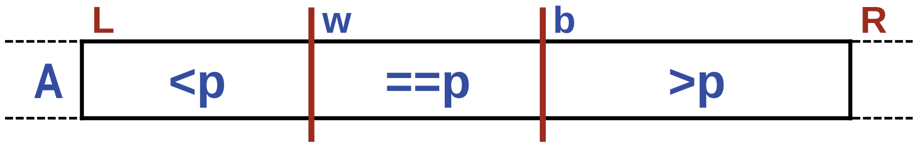
Average-Case Linear-Time Algorithm



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    while (R - L) > 1:
        p = value-of-pivot
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        # Go-on-to "<p" or ">p" region if j-th smallest there; else return p.
        if j < w: R = w
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```
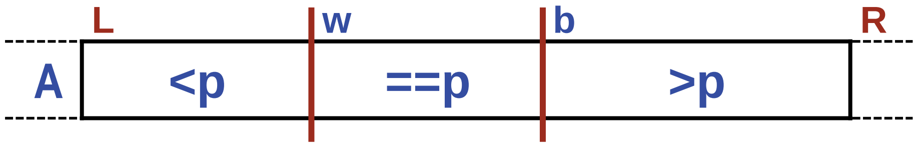

Q. Where was j ever updated?

Average-Case Linear-Time Algorithm



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        p = value-of-pivot
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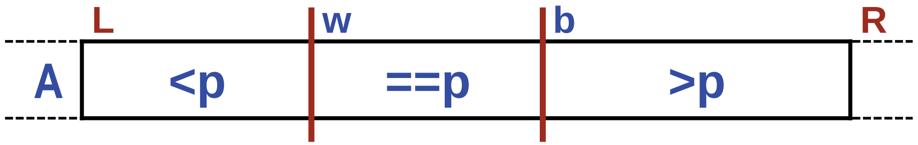
        if j < w: R = w
        elif j < b: return p
        else: L = b
    return A[j]
```

Q. Where was `j` ever updated?

A. Nowhere. Partitioning moved values so the `j`th smallest ended up in `A[j]`.

Average-Case Linear-Time Algorithm



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    """
    Given A[0..n-1], and int 0≤j<n, quick_select(A,n,j) returns the j-th smallest
    value in A[0..n-1].
    """

    L = 0;  R = n
    while (R - L) > 1:
        p = (A[L] + A[R - 1]) // 2
        ⟨body of partition⟩

        # Go-on-to “<p” or “>p” region if j-th smallest there; else return p.
        if j < w: R = w
        elif j < b: return p
        else: L = b
    return A[j]
```

Average-Case Linear-Time Algorithm

Performance: Pivots computed as (A[L]+A[R-1]) // 2

- Best case. On each iteration, pivot is (serendipitously) the median of `A[L..R-1]`, so region sizes reduced by ½. Partitioning time is linear in size.

  Total effort. $1 \cdot n + \frac{1}{2} \cdot n + \frac{1}{4} \cdot n + \ldots = 2 \cdot n$, i.e., linear in n

- Worst case. On each iteration, pivot is (serendipitously) the min or max of `A[L..R-1]`, so region sizes reduced by 1. Partitioning time is linear in size.

  Total effort. $n + (n-1) + (n-2) + \ldots + 1 = n \cdot (n-1)/2$, i.e., quadratic in n.

- Average case, i.e., summed over all permutations of values in `A[0..n-1]`.

  Total effort. Linear in n

  (offered without proof)

Bad News: quick_select can have quadratic-time performance on some arrays.

Imagine telling the widow:

> But Mrs. Jones, <u>on average</u> the code would have been fast enough to have saved your husband's life.

Goal. Linear-time performance on every array.

Performance Goal: Pivots computed as _________ in the hope that

- <u>Every case</u>.

  (1) On each iteration, region sizes reduced by constant ratio r.

  Partitioning time is linear in region size.

  Total effort for partitioning. $1 \cdot n + r \cdot n + r^2 \cdot n + r^3 \cdot n + \ldots = n/(1-r)$

  I.e., linear in *n, not counting time to compute the pivot.*

  (2) On each iteration, the cost to compute the pivot is also linear in region size.

  Thus, total effort, would be linear in *n*. In particular, even in the <u>worst-case</u>.

Performance Goal: Pivots computed as approximations to median of `A[L..R-1]`.

- <u>Every case</u>.

  (1) On each iteration, region sizes reduced by constant ratio r.

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Thus, total effort, would be linear in n. In particular, even in the <u>worst-case</u>.

Idea: Pivots computed as approximations to median of `A[L..R-1]`.

Imagine that this array, with median 61:

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
A	51	60	73	92	57	54	75	59	91	58	71	62	67	66	59	52	**61**	72	55	60	79

were laid out in a 3-high 2-D array in row major order:

51	60	73	92	57	54	75
59	91	58	71	62	67	66
59	52	**61**	72	55	60	79

The median of each column is shown in red.

Idea: Pivots computed as approximations to median of `A[L..R-1]`.

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	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
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were laid out in a 3-high 2-D array in row major order:

51	52	58	71	55	54	66
59	60	**61**	72	57	60	75
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Now, imagine that each column were sorted, so its median comes to middle row.

The median of each column is shown in red.

Idea: Pivots computed as approximations to median of `A[L..R-1]`.

Imagine that this array, with median 61:

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A	51	60	73	92	57	54	75	59	91	58	71	62	67	66	59	52	**61**	72	55	60	79

were laid out in a 3-high 2-D array in row major order:

55	51	52	54	58	66	71
57	59	60	60	**61**	75	72
62	59	91	67	73	79	92

Next, imagine that the columns were sorted by their medians. The median of the medians is shown with a green background.

The median of each column is shown in red.

Idea: Pivots computed as approximations to median of `A[L..R-1]`.

Imagine that this array, with median 61:

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
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55	51	52	54	58	66	71
57	59	60	60	**61**	75	72
62	59	91	67	73	79	92

Finally, color code the values:

pink, if ≤ median of medians

blue, if ≥ median of medians

The median of each column is shown in red.

Idea: Pivots computed as approximations to median of `A[L..R-1]`.

Imagine that this array, with median 61:

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Finally, color code the values:
pink, if ≤ median of medians
blue, if ≥ median of medians

Choose the median of medians (60) as the pivot p, and partition A.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
A	51	55	57	54	52	59	59	58	60	60	62	67	66	71	91	**61**	72	75	92	79	73

Idea: Pivots computed as approximations to median of `A[L..R-1]`.

Imagine that this array, with median 61:

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
A	51	60	73	92	57	54	75	59	91	58	71	62	67	66	59	52	**61**	72	55	60	79

were laid out in a 3-high 2-D array in row major order:

55	51	52	54	58	66	71
57	59	60	60	**61**	75	72
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We seek the median, i.e., the n/2th smallest ($n/2 = 21/2 = 10$), which falls into >p region

Choose the median of medians (60) as the pivot p, and partition A.

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Idea: Pivots computed as approximations to median of `A[L..R-1]`.

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55	51	52	54	58	66	71
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62	59	91	67	73	79	92

We seek the median, i.e., the $n/2^{th}$ smallest ($n/2 = 21/2 = 10$), which falls into >p region, eliminating at least $2/3 \cdot ½ = 1/3$ the values.

Thus, the >p region is no larger than $r = 1-1/3 = 2/3$ the size of the whole.

	0	1	2	3	4	5	6	7	8	9	**10**	11	12	13	14	15	16	17	18	19	20
A											62	67	66	71	91	**61**	72	75	92	79	73

Worst-Case Linear-Time Algorithm

Performance Goal: Pivots computed as approximations to median of `A[L..R-1]`.

- <u>Every case</u>.

  (1) On each iteration, region sizes reduced by constant ratio r.

  Partitioning time is linear in region size.

  Total effort for partitioning. $n + r \cdot n + r^2 \cdot n + r^3 \cdot n + \ldots = n/(1-r)$

  I.e., linear in *n, not counting time to compute the pivot.*

  (2) On each iteration, the cost to compute the pivot is also linear in region size.

Thus, total effort, would be linear in n. In particular, even in the <u>worst-case</u>.

Worst-Case Linear-Time Algorithm

Performance Goal: Pivots computed as median of medians of `A[L..R-1]`.

- Every case.

  (1) On each iteration, region sizes reduced by constant ratio r.

  Partitioning time is linear in region size.

  Total effort for partitioning. $n + (2/3)\cdot n + (2/3)^2\cdot n + (2/3)^3\cdot n + \ldots = n/(1-2/3) = 3\cdot n$

  ✓ I.e., linear in *n, not counting time to compute the pivot.*

  (2) On each iteration, the cost to compute the pivot is also linear in region size.

Thus, total effort, would be linear in n. In particular, even in the worst-case.

Performance Goal: Pivots computed as median of medians of `A[L..R-1]`.

- Every case.

  (1) On each iteration, region sizes reduced by constant ratio *r*.

  Partitioning time is linear in region size.

  Total effort for partitioning. $n + (2/3)\cdot n + (2/3)^2\cdot n + (2/3)^3\cdot n + \ldots = n/(1-2/3) = 3\cdot n$

  I.e., linear in *n, not counting time to compute the pivot.*

  (2) On each iteration, the cost to compute the pivot is also linear in region size.

  But how will we compute the median of medians of `A[L..R-1]`?

  Thus, total effort, would be linear in *n*. In particular, even in the worst-case.

Performance Goal: Pivots computed as median of medians of `A[L..R-1]` using recursion, i.e., apply the worst-case median algorithm to the n/3 medians of groups of 3 elements.

☞ **Consider recursion when designing an algorithm.**

Performance Goal: Pivots computed as median of medians of `A[L..R-1]` using recursion, i.e., apply the worst-case median algorithm to the n/3 medians of groups of 3 elements.

This works, but alas, there are too many groups of 3, so the total cost is super- linear.

☞ **Consider recursion when designing an algorithm.**

Performance Goal: Pivots computed as median of medians of `A[L..R-1]` using recursion, i.e., apply the worst-case median algorithm to the n/3 medians of groups of 3 elements.

This works, but alas, there are too many groups of 3, so the total cost is super- linear.

But don't loose heart. All is not lost, because …

Performance Goal: Pivots computed as median of medians of `A[L..R-1]` using recursion, i.e., apply the worst-case median algorithm to the n/5 medians of groups of 5 elements.

This works, and is linear.

Selection of a partition region eliminates at least 3/5 · ½ = 3/10 the values.

Thus, the selected region is no larger than r = 1-3/10 = 7/10 the size of the whole.

Total effort for partitioning. $n + (7/10) \cdot n + (7/10)^2 \cdot n + (7/10)^3 \cdot n + \ldots = n/(1-7/10) = 3.33 \cdot n$

In effect, the reduction ratio r shrinks slightly (from 2/3 to 3/10), but the number of groups shrinks more than enough (from $n/3$ to $n/5$) to render the total linear.

Summary:

Presented three algorithm-design principles that can serve in lieu of everyday experience:

- ☞ Consider generalizing a problem when designing an algorithm.
- ☞ Consider Divide and Conquer when designing an algorithm.
- ☞ Consider recursion when designing an algorithm.

Used the principles to derive two algorithms for finding the median as a special case of finding the `j`th smallest value in `A[0..n-1]`:

- Linear average-time performance
- Linear worst-case-time performance.