Lecture 6 CS4860 Fall 2010

Consistency of the Tableau Proof System

In Chapter II section 2, Smullyan shows that the Method of Tableaux is *consistent* in the sense that *any formula provable by Tableau is a tautology*. This result means that the proof system is safe, it won't prove anything false. The result tells us that the rules are not too strong. There might be too many rules, but if we eliminate some of them, the proof system will remain consistent because it will prove fewer formulas, possibly only a subset of the tautologies.

In the next section we will inquire about whether we have enough rules to prove every tautology. We will see that we do, and moreover, if we eliminate any of the rules, then the system will not be strong enough to prove every tautology. For this section, we focus on consistency, the "safety property" of any proof system and of any automatic proof assistant.

Theorem 1 (Consistency): Any propositional formula provable by Tableau is a tautology.

We can express this symbolically as follows. Let Tableau(FX) denote the type of all tableau proof trees with the goal FX where X is a Formula. Write Closed(T) for a tableau T if all branches are closed. Let Provable(X) be the proposition that there is proof of X, that is,

Exists T: Tableau(FX). Closed(T).

Let ClosedTableau(FX) = {T:Tableau(FX)| Closed(T)}.

Let Taut(X) be the assertion that formula X is a tautology, and let TAUT be the type of all formulas X that are tautologies and let SAT be the set of all formulas X that are satisfiable.

Here is a symbolic version of the above theorem that reads almost exactly as the informal English version.

Theorem 1 (Consistency Symbolically): For all X:Form.(Provable(X) implies Taut(X)).

Here is a variant that relates two subsets of formulas.

Corollary: For all X:Form. ClosedTableau(FX) is a subset of TAUT.

There are several other equivalent ways to state this idea. We want to pick out the one that is *easiest to prove* and easy to understand intuitively. Smullyan picks the variant that I think is easiest to prove (another reason I like his book), version 1 of the following.

1. If the origin of a tableau is satisfiable, then the tableau is open.
2. The origin of any closed tableau is unsatisfiable, that is, there is no assignment of values to variables such that the Boolean valuation of goal (origin) FX is f.

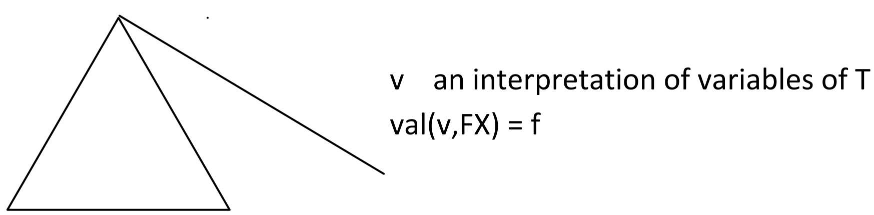
The first version means that if the goal (origin of the proof tree) is FX and if there is an assignment v of the variables of X, say Var(X), to Booleans such that the Boolean valuation val of X under v is f, i.e. val(v,X) = f in **B**, then the tableau T for FX is open.

We can abbreviate this first version as *Sat implies Open*. By propositional reasoning, the contrapositive is true, which is *not Open implies not Sat*. By definition, *not Open* is the same as *Closed*, so we have by the contrapositive*, Closed implies not Sat* and not Sat means that X is a tautology, so we have *Closed implies Taut* which is the second version listed above.

Note, if (X implies Y) is true then so is its *contrapositive*, which is (not Y implies not X). As an exercise, give a tableau proof for this fact.

Theorem 2 (Consistency) If the goal of a tableau is satisfiable, then the tableau is open, hence not a proof. Symbolically: All X:Form.((Exists v:Var(X) → B. val(v,X) = f) implies
All T: Tableau(FX). not Closed(T).)

Smullyan proves this, so for any formula X he assumes there is a Boolean valuation of the origin, say FX, which assigns the value f to X, i.e. there is an interpretation v of the variables of X such that val(v,X) = f.



closed tableau T

We show that under this assumption, we can find an open path p in the tableau T given this interpretation v; we call a path *open* under an assignment v iff every signed formula receives the value of its sign under v, that is, val(v,SY) = s where s is the Boolean value of the sign S; so sign(FY) = f and sign(TY) = t. We call the origin open if there is a v such that val(v,FX) = f.

We show that if we have built T up to some level T1 (it could be just the origin) so that it has an open path, then every extension of T1 will also be open. This is a simple fact based on the form of the rules, call it Lemma 1. Given that this is true, we can then extend T1 to the full tableau T, and it will be open. This contradicts the assumption that T is closed. So there is no valuation v of the variables that makes X false. Hence X is a tautology as we set out to show.

Lemma 1. If T is an incomplete tableau with an open path p, then any extension T' of T will also have an open path.

Exercise: Give your own proof of this fact. As a last resort, you can reproduce Smullyan's proof from page 25 where he uses the term "true tableau" where I use the term "open tableau."

Corollary: For all X:Form.(Provable(X) implies Taut(X)).

Provable(X) implies Taut(X) is the contrapositive of the theorem.

Ideas about Proofs

The key step in this proof is deciding to show the contrapositive of the normal statement, that is to prove Theorem 2. This is an appealing theorem because we have an hypothetical object v to start with and a process of building a tableau T to analyze step by step using what we know about v. Once we have a proof plan, the rest is a matter of checking details. In this case it is easy to manage the details because we only need to think about two kinds of rules that extend a partially built tableau, namely alpha (α) and beta (β) rules.