Applied Logic	**Lecture 17: Correctness and Completeness**
CS 486 Spring 2005	**Tuesday, April 5, 2005**

Now that we have introduced a proof calculus for first-order logic we have to address the usual questions again, that always come up when dealing with formal proof systems.

(1) *Is the tableau method correct? Can we be sure that a proven formula is in fact valid.*

(2) *Is it complete? Can we prove every valid formula with the tableau method?*

(3) *Is it decidable? Does it always tell us whether a formula is valid or not?*

(4) *What about compactness? What does the satisfiability of finite sets of formulas tell us?*

(5) *Are there proof strategies for building first-order tableaux that are more successful or more efficient than others?*

17.1 Correctness of First-Order Tableaux

To prove the correctness of the tableau method, one has to show that *the origin of a closed tableau is unsatisfiable* or, equivalently, that a tableau is satisfiable and cannot be closed whenever the formula at its origin is satisfiable. The basic structure of the proof is the same as the one for propositional logic, so we just formulate the key insights here.

Let $\mathcal{U}$ be an arbitrary universe and v be a first-order valuation of $E^{\mathcal{U}}$ (φ is the identity mapping).

F_1: α is true under v, if and only if α_1 and α_2 are true under v

F_2: β is true under v, if and only if at least one of β_1 and β_2 is true under v

F_3: γ is true under v, if and only if $\gamma(k)$ is true under v for every $k \in \mathcal{U}$

F_4: δ is true under v, if and only if $\delta(k)$ is true under v for at least one $k \in \mathcal{U}$

These facts follow immediately from the definition of first-order valuations on $E^{\mathcal{U}}$. As a consequence of these facts we can show the following laws about the satisfiability of sets of formulas (with parameters).

Let S be any set of formulas

G_1: If S is satisfiable and $\alpha \in S$, then $S \cup \{\alpha_1, \alpha_2\}$ is satisfiable

G_2: If S is satisfiable and $\beta \in S$, then at least one of $S \cup \{\beta_1\}$ and $S \cup \{\beta_2\}$ is satisfiable

G_3: If S is satisfiable and $\gamma \in S$, then $S \cup \{\gamma(a)\}$ is satisfiable for every parameter a

G_4: If S is satisfiable and $\delta \in S$, then $S \cup \{\delta(a)\}$ is satisfiable for every parameter a that does not occur in S

The first three laws are obvious but the last one is not, as it shows how to represent the semantical "for at least one $k \in \mathcal{U}$" by a syntactical requirement "for every new parameter a".

Proof. Let $I = (\mathcal{U}, \varphi, \iota)$ be an interpretation such that all $A \in S$ are true under I. Since $\delta \in S$, there must be at least one parameter a such that $\delta(a)$ is true under I.

Let $k = \varphi(a) \in \mathcal{U}$ and define $I' = (\mathcal{U}, \varphi', \iota)$ with $\varphi'(b) = \begin{cases} \varphi(b) & \text{if } b \text{ occurs in } S \\ u & \text{otherwise} \end{cases}$.

Then $I'(A) = I(A) = \mathrm{t}$ for all $A \in S$ and $I'(\delta(a')) = I(\delta(a)) = \mathrm{t}$ for every parameter a' that does not occur in S. Hence $S \cup \{\delta(a')\}$ is satisfiable. □

The remainder of the correctness proof is almost identical to the propositional case. We have to prove that every tableau with a satisfiable origin contains at least one satisfiable path.

Theorem 1 *Let $\mathcal{T}$ be an arbitrary tableaux whose root is satisfiable. Then there is a path θ in $\mathcal{T}$ that is uniformly satisfiable.*

Proof. We use structural induction on tableau trees.

base case: If $\mathcal{T}$ has just a single point, then let θ be the path consisting of the root of $\mathcal{T}$.

step case: Assume the statement holds for some $\mathcal{T}$ and let $\mathcal{T}_1$ be a direct extension of $\mathcal{T}$ and I be a model for the root of $\mathcal{T}_1$.

Since $\mathcal{T}$ and $\mathcal{T}_1$ have the same root there is a satisfiable path θ in $\mathcal{T}$.

We consider 5 cases (the first 3 are identical to what we had before)

(1) If $\mathcal{T}_1$ does *not* extend $\mathcal{T}$ at θ, then θ is a satisfiable path θ in $\mathcal{T}_1$.

(2) If $\mathcal{T}_1$ extends $\mathcal{T}$ at θ by some α_i, then we know that α is on θ. Thus $\theta_1 = \theta \circ \alpha_i$ is a satisfiable path in $\mathcal{T}_1$ by G_1.

(3) If $\mathcal{T}_1$ extends $\mathcal{T}$ at θ by β_1 and β_2 then β is on θ and $\theta_1 = \theta \circ \beta_1$ or $\theta_1 = \theta \circ \beta_2$ is a satisfiable path in $\mathcal{T}_1$ by G_2.

(4) If $\mathcal{T}_1$ extends $\mathcal{T}$ at θ by some $\gamma(a)$, then γ is on θ and $\theta_1 = \theta \circ \gamma(a)$ is a satisfiable path in $\mathcal{T}_1$ by G_3.

(5) If $\mathcal{T}_1$ extends $\mathcal{T}$ at θ by some $\delta(a)$ then δ is on θ and a does not occur in any of the formulas of θ. Thus $\theta_1 = \theta \circ \delta(a)$ is a satisfiable path in $\mathcal{T}_1$ by G_4.

□

17.2 Completeness

Proving the completeness of a first-order calculus gives us Gödel's famous completeness result. Gödel proved it for a slightly different proof calculus, and the proof that we will show here goes back to Beth and Hintikka. Let us briefly resume the propositional case.

The key to the completeness proof was the use of Hintikka's lemma, which states that every downward saturated set, finite or not, is satisfiable. We then showed that every open and complete path is in fact a Hintikka sequence. Putting these two things together we reasoned that the root of an open and complete tableau must be satisfiable. Thus a complete tableau for a valid formula cannot be open which means that every tableau for a valid formula will eventually close.

We will prove the first order case along these lines, but have to keep in mind that several things have changed.

- The definition of a valuation now includes quantifiers.
- The definition of Hintikka sets must take γ and δ formulas into account.
- The notion of a complete tableau needs to be adjusted, because there is now the possibility of non-terminating proof attempts.

Fortunately, we can easily make the necessary adjustments and then proceed as before. First, let us define first-order Hintikka sets. A *Hintikka Set for a universe U* is a set S of U-formulas such that for all closed U-formulas A, α, β, γ, and δ the following conditions hold.

H_0 : A atomic and $A \in S \mapsto \bar{A} \notin S$

H_1 : $\alpha \in S \mapsto \alpha_1 \in S \wedge \alpha_2 \in S$

H_2 : $\beta \in S \mapsto \beta_1 \in S \vee \beta_2 \in S$

H_3 : $\gamma \in S \mapsto \forall k \in U.\ \gamma(k) \in S$

H_4 : $\delta \in S \mapsto \exists k \in U.\ \delta(k) \in S$

The first axiom expresses the openness of S while the other four state that it is downward saturated. Note that because of axiom H_3, Hintikka sets are usually infinite, unless the universe is finite. But the proof of Hintikka's lemma that we discussed a few weeks ago, did not depend on the fact that the set is finite, so it can easily be adapted to the first-order case.

Theorem 2 (Hintikka Lemma) *Every Hintikka set is uniformly satisfiable*

Proof. Because of axiom H_0 we can define a valuation that satisfies all the atomic formulas in S.

Define $v(P(k_1, ..., k_n)) = \begin{cases} \texttt{f} & \text{if } F(P(k_1, ..., k_n)) \in S \\ \texttt{t} & \text{otherwise} \end{cases}$

To show that v satisfies every formula $Y \in S$ we proceed by structural induction on formulas, keeping in mind that the cases for γ and δ are straightforward generalizations of those for α and β.

base case: If $Y \in S$ is an atomic formula then $v[Y] = t$ by definition.

step case: Assume the the claim holds for all subformulas of Y.

- If Y is of type α then $\alpha_1,\alpha_2 \in S$, hence $v[\alpha_1] = v[\alpha_2] = t$. By definition of first-order valuations $v[Y] = t$.
- If Y is of type β then $\beta_1 \in S$ or $\beta_2 \in S$, hence $v[\beta_1] = t$ or $v[\beta_2] = t$ and thus $v[Y] = t$.
- If Y is of type γ then $\gamma(k) \in S$ for all $k \in U$, hence $v[\gamma(k)] = t$ for all k and thus $v[Y] = t$.
- If Y is of type δ then $\delta(k) \in S$ for some $k \in U$, hence $v[\delta(k)] = t$ for some k and thus $v[Y] = t$.

□

Now what about the completeness of a tableau? In the propositional case, this meant that the tableau cannot be extended any further, because all formulas have been decomposed. Since the propositional tableau method terminates after finitely many steps, this was an easy thing to define. In the first-order case, however, we have to be a bit more careful.

We know that because of γ-formulas, proofs may have infinite branches. But that is not the main problem, since Hintikka's lemma also works for infinite sets. However, not every infinite branch in a tableau is automatically a Hintikka set.

Consider for example, the formula ∃x,y.P(x,y), which is certainly not valid. Thus F(∃x,y.P(x,y)) is satisfiable and because of the correctness of the tableau method we know that every proof attempt will fail. But does *every* failing proof attempt actually give us the Hintikka set that we need to reason that F(∃x,y.P(x,y)) *must* be satisfiable?

Certainy not. Just imagine we start decomposing the main formula, which is a γ formula, over and over again. Then we can go on and on forever without ever touching the inner γ formula and we get an infinite branch that does not satisfy the third Hintikka axiom for this inner γ formula.

So our completeness proof cannot rely on an arbitrary attempt to find a tableau proof. After all, completeness only says that it must be possible to prove every valid formula correct with the tableau method but it doesn't require that *any* attempt will succeed. And the fact that we weren't able to find a proof with a not so bright approach doesn't mean that there is none at all.

However, we can design a more systematic approach that is guaranteed to find a tableau proof if there is one. And then we will show that this systematic method will always find a proof if the formula is valid.

For the systematic method we only have to worry about a treatment of γ formulas that guarantees axiom H_3, since the α, β, and δ rules make sure that the other Hintikka axioms satisfied.

Q: *How can we make sure that all γ formulas are eventually covered completely?*

Well, we have to proceed similarly to an enumeration of lists of integers. We modify the extension procedure for tableaux in a way that each γ formula, and thus every other formula as well, will be revisited on a regular basis.

A systematic procedure for proving a first-order formula X

Start with the signed formula $F(X)$ and recursively extend the tableau as follows:

- If the tableau is already closed then stop. The formula is valid.
- Otherwise select a node Y in the tableau that is of *minimal level* wrt. the still unused nodes and extend *every* open branch θ through Y as follows:
  - If Y is α extend θ to $\theta\cup\{\alpha_1,\alpha_2\}$.
  - If Y is β, extend θ to two branches $\theta\cup\{\beta_1\}$ and $\theta\cup\{\beta_2\}$.
  - If Y is γ, extend θ to $\theta\cup\{\gamma(a),\gamma\}$, where a is the first parameter that is not on θ.
  - If Y is δ, extend θ to $\theta\cup\{\delta(a)\}$, where a is the first parameter that does not occur in the tableau tree.

Thus the procedure always copies a γ formula to the end of a branch when it is being considered. This way we make sure that it is considered over and over again, but that all the other formulas on the branch are decomposed before that. Thus in the end all the formulas are being used, because we have only denumerably many parameters. This method is certainly not the most efficient one, but it works.

Using the systematic procedure we can give a new definition of complete tableau. A systematic tableau is called *finished*, if it is either infinite or finite and cannot be extended any further. With this definition we immediately get the following result.

Lemma 3 *In every finished systematic tableau, every open branch is a Hintikka sequence.*

A detailed proof for this lemma would have to show by structural induction that the systematic method does in fact cover all formulas as required in the Hintikka axioms. Together with Hintikka's lemma we get.

Corollary 4 *In every finished systematic tableau, every open branch is uniformly satisfiable.*

The completeness theorem is now an immediate consequence as before.

Theorem 5 (Completeness theorem for first-order logic)

If a first-order formula X is valid, then X is provable. Furthermore the systematic tableau method will construct a closed tableau for $F(X)$ after finitely many steps.

The first statement follows from the above corollary by contraposition and the fact that the systematic tableau method always "constructs" a finished tableau. As for the second, a closed tableau can only have finite branches, which – according to König's lemma – means that it must be finite.

Note that correctness and completeness is preserved again if we require an *atomically closed tableau*, i.e. a tableau where branches only close if there is an atomc formula and its conjugate. Correctness follows from the fact that an atomically closed tableau is certainly a closed tableau, while the systematic tableau method makes sure that we construct a Hintikka sequence if the tableau does not close (which is the case if it does not close atomically). Hintikka's lemma thus implies

Corollary 6

If a first-order formula X is valid, then X the there is an atomically closed tableau for $F(X)$.

The corollary also has a second important consequence that will be relevant for the compactness of first-order logic.

Theorem 7 (Löwenheim theorem for first-order logic)

If a first-order formula X is satisfiable, then it is satisfiable in a denumerable domain.

The proof for this theorem is based on the observation that the systematic tableau method uses only denumerably many parameters to build a Hintikka sequence if the tableau doesn't close. Since a tableau with a satisfiable formula at its root cannot close, it must contain an open branch θ with at most denumerably many parameters. As this branch is uniformly satisfiable it satisfies X in a denumerable domain (the subset of the domain U that represents the set of parameters on θ).