CS 684 Algorithmic Game Theory	**December 5, 2005**
Instructor: Éva Tardos	**Scribe: Sameer Pai**

1 Current Class Events

Problem Set 3 solutions are available on CMS as of today. The class is almost completely caught up with class notes. Everything is on track except for the final projects, which have not shown up as yet. In today's class we will compare what we have learned about truthful games (see November 28-30, 2005 lecture notes) to Nash games.

2 Comparison Between Truthful and Nash Auction Games

We start with the example of selling one item with n possible buyers. There are two types of games that can be played:

2.1 Truthful Game

In the realm of truthful games, we can have a Vickery Auction. This is an auction where the buyers secretly announce their values to the auctioneer. The values are ordered in decreasing order, $v_1 \geq v_2 \geq ... \geq v_n$, by the auctioneer. Then, the auctioneer sells the item to the buyer with the highest value, v_1, at a price equal to the second highest value, v_2.

2.2 Nash Game

In the realm of Nash games, we have a First Price Auction. This is an auction where the buyers secretly announce their bids, player i announces bid p_i. The auctioneer sells to player with maximum p_i at his claimed p_i value (see figure 1).

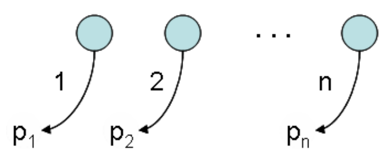

Figure 1: Example Nash Game: First Price Auction

This game begs the question: As a buyer, how do you know what is a good price to pay (bid) in this first price auction?

The issue hinges upon what person i knows about the others playing the game (his neighbors).

- Option 1: *Assume Full Information Known*; a Game in which the value for each player $v_1, v_2, ..., v_n$ is known by everyone playing the game (e.g. model of airline ticket purchasing as used in Economics).
- Alternatives: *Assume Random Information Known*; a Game in which the probability distributions of all values, v_i, is known everyone playing the game. Here we assume that each v_i is independent. We note that this is a general case, and that Option 1 is a special case of this (this is when v_is are deterministic).

Next, we determine the concept of a Nash equilibrium assuming Option 1. Consider figure 2. There are many concepts of a Nash here, however, we focus on intuitively the most natural one.

We see here that if the buyer with the highest value, v_1, bids v_2, so that $p_1 = v_2$, no other buyer can beat him. This is true since other buyer's values are not big enough to bid above the buyer with the highest value, and if they bid below this buyer, they lose. The player with value v_1 has no incentive to change his bid, since he already wins by bidding $p_1 = v_2$; bidding higher only decreases his benefit, and bidding lower could prevent him from winning the item (since the buyer with the next highest value would beat him). For a Nash equilibrium, it must also be true that the buyer with the second highest value, v_2, must bid his value, so $p_2 = v_2$. This is true because, if the buyer with the second highest value says something lower, then the buyer with the highest value can drop his bid, and still win. Finally, at Nash, the other buyers can bid whatever they wish, since they cannot effect the outcome (the strategy of the buyer with the highest value, is only dependent on the strategy of the player with the second highest value). Thus, in this Nash, we have $p_1 = p_2 = v_2$ and all other buyers bid whatever they wish.

Note that there is something a bit awkward with the Nash presented above. Mainly, it is unclear which buyer, the one with the highest value or the one with the second highest value, will win. This issue must be resolved by tie-breaking, and it is unclear how we can justify the buyer with value v_1 winning over the buyer with value v_2. If the buyer with the highest value bids a tiny amount over v_2, our solution is not exactly a Nash anymore.

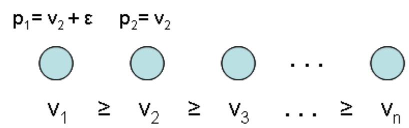

Figure 2: Example First Price Auction With ϵ-Nash; Nodes Represent Buyers

There are two ways in which we can get around this problem:

First, we can consider the concept of an ϵ-Nash, where buyers will only deviate if there is some gain greater than ϵ. This allows the buyer with the highest value, v_1 to bid $p_1 = v_2 + \epsilon$ and win (see figure 2). We can take the limit as $\epsilon \to \infty$, to get back our original Nash solution; now the player with the highest value wins. This gets around our issue of tie-breaking.

Second, we can determine the concept of a Nash equilibrium assuming the Alternative to Option 1 shown above. We assume here that we are given a (non-deterministic) probability distribution by which a buyer i select their value, v_i, for the item. For this assumption the unique Nash is: Given buyer i draws value v_i (kept private), then his bid should be $p_i = \mathrm{E}[max_{j \neq i} v_i \mid v_i \text{ is max value}]$. We further assume that the event we condition on has some non-zero probability of occurring (required for uniqueness). Note that this is a nice analogue of what happened assuming Option 1, above; everyone bids what they believe is the second highest price. We also note that calculating p_i in this way is usually a difficult calculus computation [M81].

We can give an example of this, to offer some intuition to what a Nash equilibrium under this assumption actually is: Suppose that every buyer i draws a value uniformly independently at random from the interval [0,1] (see figure 3). Then, if .99 is drawn by a buyer, then this buyer probably has an excellent chance of winning. Given the event that this value is the highest, the buyer's bid should be close to his expectation of the second highest price (perhaps .9). Similarly, if .3 is drawn by the buyer, then he does not have a great chance of winning (note that in the expression below, this is taken into account by conditioning on the event that this value drawn is indeed the maximum value). In this case, the player should still bid close to what he believes is (his expectation of) the next highest price, if he would like win and get the most benefit he could.

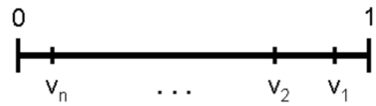

Figure 3: Uniform Independent Draws Of Values From [0,1] Interval

3 Comparison Between Truthful and Nash MST and Path Games

3.1 Minimum Spanning Tree (MST) Games

We will examine a MST game: In this game, we have a graph G = (V,E). The edges e_i in the graph represent users. There is a coordinator interested in buying and using a Minimum Spanning Tree (MST), T, in G (see figure 4). We suppose user, e_i, has a private cost c_{e_i} for his edge, e_i, being used. This cost is also the associated edge weight. This is the true cost of the edge being used, and the user will not allow the edge to be used for less.

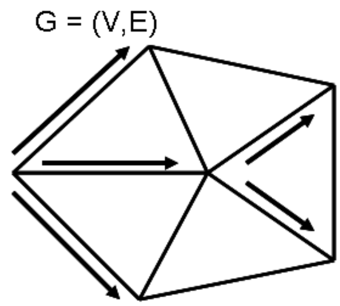

Figure 4: A Spanning Tree (containing all vertices) In G

3.1.1 Truthful MST Game

Next, we examine a truthful game similar to the one presented in the lecture notes of November 28^{th}, 2005. In that game, a coordinator wanted to buy an source-destination path in a graph G = (V,E). In this game, the coordinator is interested in buying and using a MST, T, in G. We suppose that each user places a bid equal to its cost c_{e_i}. The coordinator is to select a MST, T, and pay each $e_i \in T$ (this could be more than the cost). The Vickrey-Clark-Groves (VCG) mechanism, studied in class on November 28^{th}, 2005, allows the coordinator buy an MST in such a way so that the users have no incentive to lie about their costs. As found in that class, the following payments result:

$$\text{Payment to user } e_i \text{ is } p_{e_i} = c_{e_i} + (MST^{e_i} - MST)$$
$$MST = \text{minimum spanning tree cost}$$
$$MST^{e_i} = \text{minimum spanning tree cost in } G - e_i$$

As found last week, this payment, which exceeds the actual cost, enforces that a user will not lie when reporting their cost.

3.1.2 Nash MST Game

Alternatively, we examine a Nash Game; the First Price Auction with the assumption of full information (Option 1). In this game, the coordinator is interested in buying and using a MST, T, in G. We suppose that each user places a bid and we assume everyone knows everyone else's cost c_{e_i}. The coordinator is to select a MST, T, and pay each $e_i \in T$ (this could be more than the cost). To avoid a tie-breaking problem, we will use the concept of an ϵ-Nash as we defined above. Examining figure 5, we see that edges, will tend to deviate from their chosen strategy (a bid amount), if they can gain greater benefit by doing so. We see how edges could potentially deviate in figure 5 (we show a potential strategy (bid equal to cost) by the user, followed by an arrow, followed by his a newly selected strategy). However, we would like to know what bid will the edges settle on and what are the payments that the coordinator will have to make? In other words, what is the Nash equilibrium for this game?

Theorem 1 ([KKT05]) *Making a payment to user e_i, of $p_{e_i} = c_{e_i} + (MST^{e_i} - MST)$ is Nash.*

Proof. The proof comes from the theory of algorithms (see [CLR90]). We assume that all costs are different. We first note that if an edge, e_i, is removed from a spanning tree, the the rest of the edges will remain, and another candidate edge contained in a cycle with edge e_i will replace it. Next, we know that an edge, e_i, is in a MST if and only if:

- 1: e_i is the minimum edge in a cut
- 2: e_i is not the most expensive edge in a cycle

Note that the second statement implies that an edge e_i will remain in the MST if and only if it is not the most expensive edge in any cycle it is a part of. Thus, barring this, user e_i will bid the maximum possible amount such that he remains in the MST. This turns out to the payment to this user as defined above. Otherwise, user e_i will not be in the MST and will have zero benefit. ■

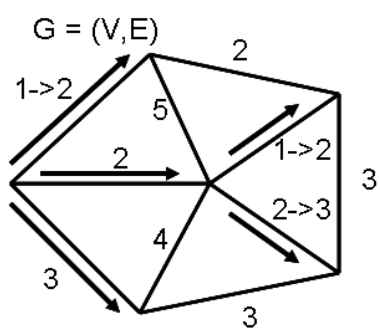

Figure 5: A Minimum Spanning Tree (containing all vertices) In G, Where Users Play Nash

3.1.3 MST Game Remarks

We note, from the results of Theorem 1, that the VCG mechanism payments are the same as those for the First Price Auction. In fact, this result is true if and only if the allowed set system is a matroid [KKT05].

3.2 Path Games

The path game was presented in the lecture notes of November 28^{th}, 2005. We will examine this path game: In this game, we have a graph G = (V,E). The edges e_i in the graph represent users. There is a coordinator interested in buying and using the shortest distinguished source (s) to distinguished destination (t) path, P, in G (see figure 6). We suppose user, e_i, has a private cost c_{e_i} for his edge, e_i, being used. This cost is also the associated edge weight. This is the true cost of the edge being used, and the user will not allow the edge to be used for less.

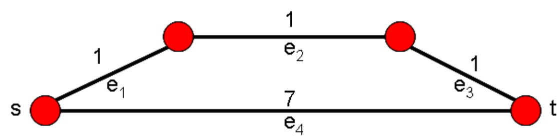

Figure 6: A Figure Of A Path Game

3.2.1 Truthful Path Game

In this game, the coordinator wants to buy the shortest source-destination (s-t) path in a graph G = (V,E). We suppose that each user places a bid equal to its cost c_{e_i}. The coordinator is to select a path, P, and pay each $e_i \in P$ (this could be more than the cost). The Vickrey-Clark-Groves (VCG) mechanism, studied in class on November 28^{th}, 2005, allows the coordinator buy a path in such a way so that the users have no incentive to lie about their costs. As found in that class, the following payments result:

$$\text{Payment to user } e_i \text{ is } p_{e_i} = c_{e_i} + (SP^{e_i} - SP)$$
$$SP = \text{cost of shortest path from s to t}$$
$$SP^{e_i} = \text{cost of shortest path in } G - e_i \text{ from s to t}$$

As an example, see figure 6. Here, we determined that the shortest path consists of the edges e_1, e_2, e_3. The coordinator would have to be pay each of these edges 5 in order to make this game truthful under the VCG mechanism.

3.2.2 Nash Path Game

Alternatively, we examine a Nash Game; the First Price Auction with the assumption of full information (Option 1). In this game, the coordinator wants to buy the shortest source-destination (s-t) path in a graph G = (V,E). We suppose that each user places a bid and we assume everyone knows everyone else's cost c_{e_i}. The coordinator is to select a path, P, and pay each $e_i \in P$ (this could be more than the cost). To avoid a tie-breaking problem, we will use the concept of an ϵ-Nash as we defined above.

We would like to know what the users will bid and therefore what the coordinator will pay each edge at Nash. Using our knowledge of a Nash, it is clear that the Nash equilibrium involves paying each edge in the shortest path, P,(each $e_i \in P$)in such a way that $p_{e_i} > c_{e_i}$ and so that the sum of p_{e_i}, for all $e_i \in P$ is equal to the cost of next shortest path. In figure 6, this means that we pay each edge e_1, e_2, e_3 more that 1 each and so that $p_{e_1} + p_{e_2} + p_{e_3} = 7$. Here we can once again use the notion of an ϵ-Nash to avoid tie-breaking, and to naturally choose the path consisting of edges e_1, e_2, e_3 as being the shortest.

We can also have some awkward looking Nash equilibria in path games. Examining figure 7 we see that the shortest s-t path consists of choosing edges e_1 and e_2 with total actual cost of 2. However, if these edges lie and bid 7 as do edges e_3 and e_4, the the shortest path chosen consists of edge e_6 at a cost of 6. Another example is as follows: Suppose the edges e_3 and e_4 bid 3 each such that the path containing edges e_3 and e_4 would cost a total of 6. We assume here that edges e_1 and e_2 both bid 6. This brings the total cost of the path containing e_1 and e_2 to 12, however, neither edge can change the fact that their total is too high individually. Then, we see that the path containing e_3 and e_4 could potentially win. This example is depicted in figure 7.

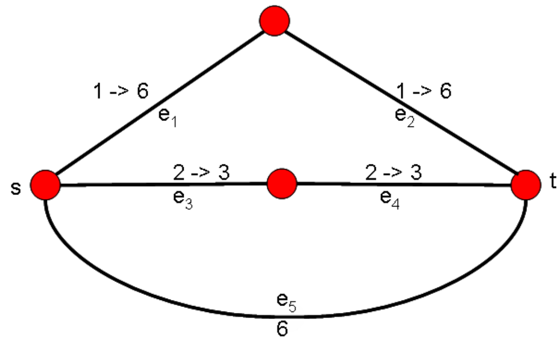

Figure 7: A Path Game With Awkward Nash Equilibria Possible

Another natural question is what is the price of the cheapest Nash? This is the best Nash equilibrium in a path game. It is important to understand what the cost of the cheapest path is. This question is further pursued in [KKT05].

3.2.3 Path Game Remarks

We note that, in general, the VCG mechanism payments are not the same as those for the First Price Auction in path games. Also, there can be awkward Nash equilibria in certain path games.

4 Some Final Words

Combining truthful games and Nash games, in some useful way, seems to be an open area of research. The authors of [KKT05] are one of the first to explore this area. Finding the most useful aspects of Nash games and combining them with the most useful aspects of truthful games seems to be a good direction for this research to go in.

References

[KKT05] Karlin, A., Kempe, D., and Tamir, T. Beyond VCG: Frugality of Truthful Mechanisms. 46^{th} Annual IEEE Symposium on Foundations of Computer Science 2005 (FOCS 2005) Proceedings: 615–626.

[M81] Myerson, R.B. “Optimal Auction Design.” Mathematics of Operations Research, 6(1), pp. 58. (1981)

[CLR90] Cormen, T.H., Leiserson, C.E., and Rivest R.L. Introduction to Algorithms. MIT Press: Cambridge, Mass.; McGraw-Hill Book Company: Boston, (1990).