CS 684 Algorithmic Game Theory	April 7, 2004
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For the last two lectures, we consider cost-sharing with *convex* cost function $C(q)$ where $q_1, \ldots, q_k$ are users' desired quantities of goods/services and $q = \sum q_i$. Recall that with convex cost functions, the total cost increases faster than linear and users are worse-off when there are more of them.

Today, we will consider cost-sharing with *concave* function.

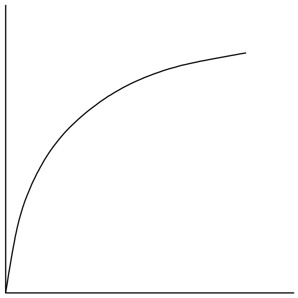


Figure 1: An example of a concave cost function.

In this scenario, users actually like being together because of economy of scale. In addition, users have utility for being included. We will assume that users are inflexible and will only be satisfied if they receive their desired quantities.

Formally, the utility of user $i = \begin{matrix} u_i & \text{if user } i \text{ gets } q_i \\ 0 & \text{otherwise} \end{matrix}$

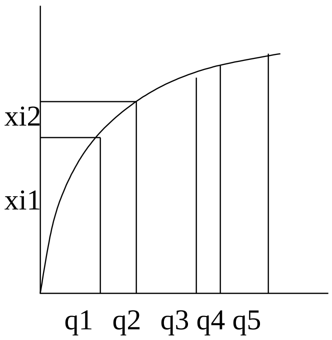
That is, unless the quality is good enough, they don't want to participate.

Recall that we let ξ_i be the cost-share (monotone) for user i. We want to have our cost-sharing scheme to be in the core. That is, we would like to cover the cost $\sum \xi_i = C(N)$ where $C(S)$ is the cost for a set $S \subseteq N$. In addition, users should not be able to do better by splitting into groups, i.e. $\sum_{i \in S} \xi_i \leq C(S)$ all subsets S.

Theorem 1 *Supposer there are k users requesting $q_1, \ldots, q_k$. Let $q = \sum q_i$ and let the cost $C(q)$ be concave. Then cost sharing in the core is possible.*

Proof: Will follow later. $\square$

Consider the following payment scheme where each user pays the additional cost to the system resulting from her or her request.



Making player i pay $\xi_i = C(q_1 + \ldots + q_i) - C(q_1 + \ldots + q_{i-1})$ is in the core. Thus, the Shapley value is in the core.

We will now consider the property of set function $C(S)$ that implies the Shapley value is in the core.

- *Submodularity:* For all sets A and B,

$$C(A) + C(B) \quad \geq \quad C(A \cup B) + C(A \cap B) \tag{1}$$

  (Submodularity is the set system analog of concavity.)

- Equivalently, for all i and sets $A \subseteq B$,

$$C(A + i) - C(A) \quad \geq \quad C(B + i) - C(B) \tag{2}$$

  That is, it is better for user i to arrive later.

Lemma 2 *The two inequalities are equivalent.*

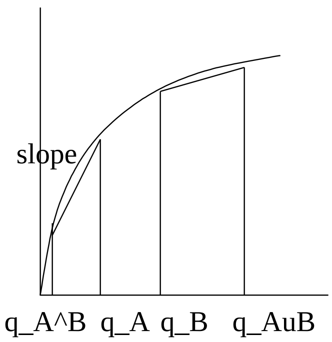
Proof: (1) $\Rightarrow$ (2): Take set $(A + i)$ and B. The only relevent case is when $i \notin B$.
By (1), $C(A + i) + C(B) \geq C(A) + C(B + i)$.

(2) $\Rightarrow$ (1): By induction on $|A \setminus B|$.
(2) is the condition when $|A \setminus B| = 1$.
(The rest is not hard to do.) $\square$

Theorem 3 *If C is submodular, then cost shares $\xi_i = C(1, \ldots, i) - C(1, \ldots, i - 1)$ is in the core (and also Shapley value).*

Proof: See later. $\square$

Note: If $\text{Cost}(S) = C(\sum_{i \in S} q_i)$ and C is concave, then $\text{Cost}(S)$ is submodular.



Consider sets A and B.
$q_A = \sum_{i \in A} q_i$ and $q_B, q_{A \cup B}, q_{A \cap B}$ are similarly defined.
By concavity, the slope $\frac{C(q_A) - C(q_{A \cap B})}{q_A - q_{A \cap B}} \geq \frac{C(q_{A \cup B}) - C(q_B)}{q_{A \cup B} - q_B}$.
Note that $q_A - q_{A \cap B} = q_{A \setminus B} = q_{A \cup B} - q_B$.
Rearranging the terms and we get the desired result.

Goal: A game such that

- users tell utility
- design solution share cost
- user may leave if $\xi_i > u_i$
- then new round

By revelation principle, we may assume that users' role is to tell u_i to the algorithm. Note, however, that people may not know their utility.

Question: Are users always truthful? Do they announce true u_i or do they leave when $\xi_i > u_i$?

Well, it depends on the algorithm!

The following theorem gives a property of cost-share and how to deal with leaving players that causes truthful behavior.

Theorem 4 *If the share $\xi_i(S)$ for set S of players is monotone increasing as S decreases, then rational users are truthful. That is, for $i \in A \subset B$, $\xi_i(A) \geq \xi_i(B)$ (**cross-monotone**).*

Theorem 5 *If $C(S)$ is submodular then $\xi_i = C(1, \ldots, i) - C(1, \ldots, i-1)$ and the Shapley value are cross-monotone.*

Proof: We need $\xi_i(A) \geq \xi_i(B)$ for $A \subset B$.
Let $S_i = \{1, \ldots, i\}$.
$\xi_i(A) = C(S_i \cap A) - C(S_{i-1} \cap A)$
$\xi_i(B) = C(S_i \cap B) - C(S_{i-1} \cap B)$
This follows for submodularity with sets $S_i \cap A$ and $S_{i-1} \cap B$. □

Claim 6 *Cross-monotone cost sharing is in the core.*

Proof: $\sum_{i \in S} \xi_i(N) \leq \sum_{i \in S} \xi_i(S) = C(S)$
The inequality follows from cross-monotonicity and the equality from cost sharing. □