CS 684 Algorithmic Game Theory	March 8, 2004
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Solution Concepts and Nash Equilibria

1 Introduction

Consider a game $G = (\mathcal{S}, \pi)$ where $\mathcal{S}$ is the set of strategy and π is the set of payoff.

We are interested in the outcome of the game $\mathcal{O}(G) \subseteq \mathcal{S}$. Note that different set of players can lead to different outcome.

Today we are interested in when $\mathcal{O}(G) = Nash(G)$. We will also consider $max\, Nash(G)$ and $min\, Nash(G)$.

Example: Cournot Duopoly/FIFO Queuing

In this game, users share a network link. Assume the game consists only of a few big players who follow selfish routing.

When $n = 2$, $\mathcal{S} = [0, \infty], \pi_i(\mathcal{S}) = S_i(1 - S_i - S_j)$ and $Nash(G) = \{(\frac{1}{3}, \frac{1}{3})\}$.

This is inefficient since they would be better off at $(\frac{1}{4}, \frac{1}{4})$.

Next, we will consider undominated strategies $\mathcal{U}(G)$. The idea behind this is that some strategies are clearly not good to play since they are dominated.

Example: Prisoner's Dilemma

	C	D
C	1,1	-1,2
D	2,-1	0,0

Since C is strictly dominated by D, $\mathcal{U}(G) = \{(D, D)\}$. This is also Nash.

Note: $Nash(G) \subseteq \mathcal{U}(G)$. That is, we don't want to play stupid strategies.

2 Selection Criteria

(0) Well defined

(1) Obvious outcome

(2) Rationality (therefore predictable)

(3) Learning (Learnable $\rightarrow$ we can get to Nash)
- distributed algorithm

(4) Myerson Criteria

3 Problems with Nash

(0) **Well defined**

- Nonexistence
  Solution: mixing, convexification
- Multiple equilibria (Strong implementation)
  This one we can't really get rid of. Consider the Schelling example from *Strategy of Conflict*.
  You want to meet a friend in New York. You don't know where to meet. Where should you go?
  There are lots of Nash equilibria, all equally good (and all equally useless), that is when the two people are at the same place.
  Solution Idea: "Refinements." However, most proposed solutions are very specific to the considered problems and generalization does not appear possible.
  What about generating a game matrix at random?
  Then $\mathbb{E}(\#Nash)$ is exponential in everything. For example, if we have 6 players and 6 strategies, then $\mathbb{E}(\#Nash) \approx 5 \times 10^6$ (Mclennan, University of Minnesota).

(1) **Obvious outcome**

"If it is so obvious, why didn't people think about it?"

"Obviousness is to the eye of the beholder."

(2) **Rationality**

Osbourn & Rubenstein: Outcome is *Rationalizable set* (for most games, this is everything). That is, we are in the situation where "I think what I'll do is good for what you'll do which will be good for what I'll do which ..." and this can lead to a very large set of possible outcome.

(3) **Learning**

We want a method that players can follow and converge to Nash.

- No polynomial time algorithm for Nash is known for general problems.
- Fudenberg & Levine, 1993: "Rational learning leads to Nash equilibrium" (with some technical restrictions). However, Nachbar (1997) showed that these technical restrictions are impossible since they require solving doubly exponential problems in every stage.

Examples of Simple Learning

- **Best Reply Dymamics**
  Assume the other players won't change, what should you do? Same for other players.
  Consider the following game:

	l	c	r
t	10,15	0,0	15,10
m	0,0	1,1	0,0
b	15,10	0,0	15,10

Nash exists in the middle. However, if the players start at top left, this will cycle forever.
Let $\mathcal{BR}(G)$ be where players end up if the game converges. In the case of FIFO Queuing, the two players will converge.

Aside:

	l	r
t	10,10	10,9
b	$9, -2^{1}0$	8,8

$\rightarrow Nash = (10, 10)$ but the column player will always play right. So Nash is not Robust

- **Adaptive Dynamics**

  Each player predicts what the other players will play based on history with finite memory. Let $P(h^t)$ be the prediction (must be in the set of previous plays). Each player uses best reply to $P(h^t)$.

  Consider a fictitious play where each player find probablility distribution from past play then gives best reply. This was thought to lead to Nash until Shapley provided a counter example based on Rock/Paper/Scissors.

  Example: Prisoner & Altruist

  Let A be the row player (Altruist) and B be the column player (Prisoner).

	C	D
C	1,1	-1,2
D	0,-1	0,0

  In the first iteration, $\mathcal{U}(G) = \{(C, D), (D, D)\}$.

  Now A knows that B will defect.

  So in the second iteration, A will defect too, so $\mathcal{U}^2(G) = \{(D, D)\}$ is the only rationalizable strategy.

  Theorem 1 *(Miligrom & Roberts, 1992) Let $\mathcal{U}^\infty$ be the serially undominated set. Adaptive learning leads to $\mathcal{U}^\infty$.*

  An extension by Friedman & Shenker (1999) shows that no assumptions are needed for the previous theorem to hold. (Players do not need to know the matrix, etc).

- **Asynchronous Learning**

  Consider the same game as before: (A is the row player and B is the column player)

	C	D
C	1,1	-1,2
D	0,-1	0,0

  What if players play at different speed, will this converge?

  Suppose A is very fast (updating every second) and B is very slow (updating every minute). Also assume that they do not know the payoff matrix.

  Since A can update every second, A can adjust to what B plays. That is,

  If B plays C, A will play C.

  If B plays D, A will play D.

  What happens?

  B will see that when he plays C he gets approximately 1 and when he plays D he gets approximately 0.

  Thus, this will converge to (C, C) – Stackelberg equilibria.

  Generally, this will not converge to Nash.

(4) Myerson Criteria

"What else should I use?"