CS 684 Algorithmic Game Theory	Scribe: Chandrashekhar Nagarajan
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1 Market Equilibria

Market life is actually a potential game played by real persons with parameters similar to the potential game parameters. However the decisions taken by the players seek to maximize their own private utility. We will consider some simple market games in this lecture and try to analyze their equilibria if exists.

1.1 Simple example

Let us look at a very simple market game. This is a seller-location game where there is demand for only one product, say ice-cream. Consider a scenario where the demand is from the buyers who are uniformly distributed on a line segment from 0 to 1 on the real line $\mathbf{R}$. Also assume that the buyers will get their ice-creams from the nearest seller. We will analyze the game for k sellers where the objective of each seller is to place his shop on $[0, 1]$ so as to maximize his profit i.e the number of customers served by him.

Case 1. $k = 1$ This is the case of monopoly. The seller can place his shop anywhere on $[0, 1]$ and the buyers have no other options but to get their ice-creams from him. So any location is a Nash as he has no incentive to change his current location.

Case 2. $k = 2$ In this case there are two sellers who want locate their shops on $[0, 1]$ so as to maximize the number of customers served by each of them. We will discuss how the Nash solution in this case looks like if it exists. Suppose in a solution if the location of shops of two sellers are separated by some distance then this solution cannot be Nash as each seller can increase the number of customers by moving closer to the other seller. Also the solution where the two sellers are at same location other than 0.5 can't be a Nash as any seller can move closer to 0.5 to increase his number of customers. So the only Nash solution is when both sellers are at 0.5 on the line $[0, 1]$. So the only deterministic Nash is when the sellers are located on top of each other at 0.5

Case 3. $k = 3$ In this case where there are 3 sellers if all the 3 sellers are not located on each other then the single seller who is not in the middle of other sellers will tend to move toward other sellers to increase his share of customers. So there cannot be a deterministic Nash in this scenario. Even if all three sellers are top of each other at some location other than 0.5 then any one seller can move towards 0.5 and increase his share of customers from one-third to some fraction greater than half. If all customers are at 0.5 any single seller can move bit from 0.5 to increase his share from one-third to close to half of the customers. So in this case no deterministic Nash solution exists as we have exhausted all possibilities.

1.2 Same example with prices

In the previous example we considered the price of an ice-cream is the same irrespective of the seller. In this model we allow the sellers to determine their price for the product. One problem in this model is that the buyers may have a trade-off between cheapness of the product (ice-cream) and the nearness of its availability. Another problem is that the price may affect the quantity of ice creams bought by the buyer. We may also have to think about how to determine the social goodness in this model. One way is to consider social goodness as more goods produced for lesser price. So the seller will have to strike a balance between minimizing the price to attract the buyers and also maximize his profit.

1.3 A market game with prices (Vetta 2002)

Input :

- n markets (M).
- k suppliers.
- L_j, a set of possible locations for each supplier j.
- A bipartite graph on set of markets and set of possible locations with cost on edge $c_{ij} \geq 0$ which is the cost for serving market i from location j.
- p_i, maximum price that can be paid by market i to buy the product.

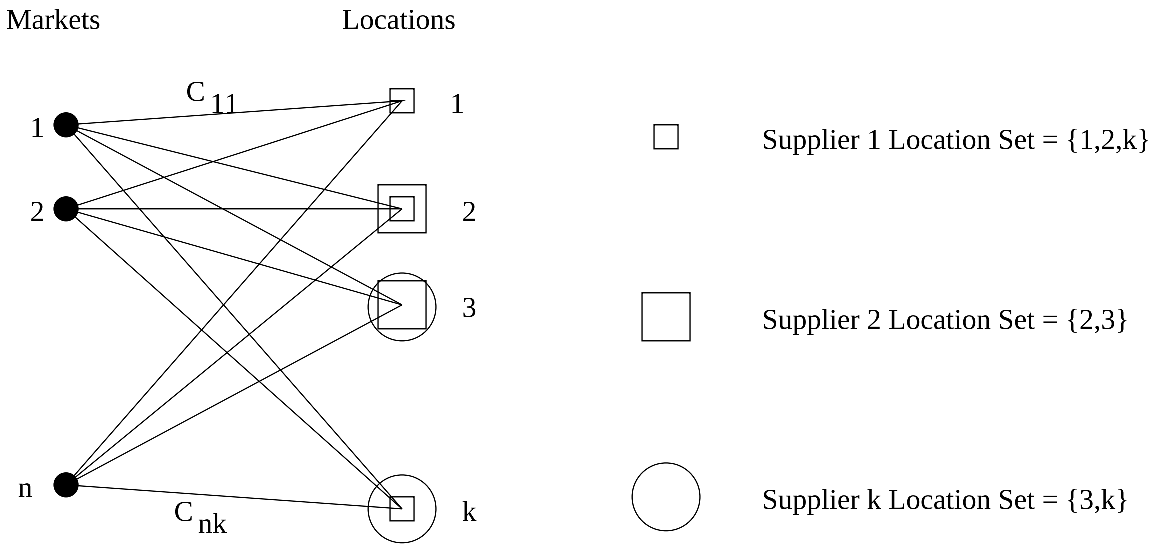

Figure 1: Market - Supplier game.

Let us consider a special case of this model where the locations of the suppliers are fixed. Let L' be the set of locations from which the markets can be served i.e the locations where suppliers are available. Let us assume that π_{ij} be the price determined by the supplier at location $j \in L'$ to the market i.

Let us analyze the game in the perspective of markets and the suppliers. The market i tries to get their goods from the cheapest available location $j \in L'$. Let us call this cheapest location $j = \sigma(i)$.

$$\sigma(i) = arg\left(\min_{j \in L'}(\pi_{ij})\right)$$

The supplier at a location $j \in L'$ tries to lower the price π_{ij} at market i below other locations price $\pi_{ij'}, j \neq j'; j' \in L'$ provided $\pi_{ij} \geq c_{ij}$. The profit he makes at market i is $\pi_{ij} - c_{ij}$. If $\pi_{ij} < c_{ij}$ then the supplier at j is at a loss selling goods at market i. A supplier at location j will be able to serve the market i over other locations in L' if and only if his price is the minimum of all prices the locations that serve the market i. Also note that to be able to lower the price below others and at the same time get profit, the supplier at location j have the cost c_{ij} less than other costs $c_{ij'} j' \in L'$. So

$$\sigma(i) = arg\left(\min_{j \in L'}(c_{ij})\right)$$

The supplier at location $\sigma(i)$ will assign a price equal to second minimum of $c_{ij}; j \in L'$ or p_i whichever is smaller if $c_{i\sigma(i)} \leq p_i$[1]. So

$$\pi_{i\sigma(i)} = \min\left(p_i, \min_{\substack{j \in L' \\ j \neq \sigma(i)}} c_{ij}\right)$$

Theorem 1 *This game is a potential game.*

Proof : Let M' be the set of markets that are assigned to some location in L'. Let us define a function

$$\Phi = \sum_{i \in M'} c_{i\sigma(i)} + \sum_{i \notin M'} p_i$$

This potential function gives some idea on the quality of the solution. This potential function is actually the social value function which captures the actual cost of the situation. To show that this is indeed a potential function we have to show that a single selfish move of a particular supplier changes the potential function by exactly the amount of change in the supplier's revenue. Consider a supplier changing his location from j to j'. Let the potential function change from Φ to Φ'. Let $\sigma'(i)$ be the new locations associated with market i after the location change of a supplier from j to j'. We abuse the notation $c_{i\sigma'(i)}$ and $c_{i\sigma'(i)}$ to also denote p_i if the corresponding market i is not assigned at all to any location.

Divide the markets M into four disjoint sets S_1, S_2, S_3, S_4 defined as follows.

$$S_1 = \{i : \sigma(i) = j \textit{ and } \sigma'(i) = j'\}$$
$$S_2 = \{i : \sigma(i) = j \textit{ and } \sigma'(i) \neq j'\}$$
$$S_3 = \{i : \sigma(i) \neq j \textit{ and } \sigma'(i) = j'\}$$
$$S_4 = \{i : \sigma(i) \neq j \textit{ and } \sigma'(i) \neq j'\}$$

Consider the change in profit P_i of the supplier corresponding to each of these set of markets S_i.

$$P_1 = \sum_{i \in S_1} (c_{i\sigma(i)} - c_{i\sigma'(i)})$$

[1] If $c_{i\sigma i} > p_i$ then the supplier won't get any profit by selling to market i as the market will not pay a price $> p_i$

$$P_2 = -\sum_{i \in S_2} (c_{i\sigma'(i)} - c_{i\sigma(i)})$$

$$P_3 = \sum_{i \in S_3} (c_{i\sigma(i)} - c_{i\sigma'(i)})$$

$$P_4 = \sum_{i \in S_4} (c_{i\sigma(i)} - c_{i\sigma'(i)}) = 0$$

Each of the profit P_i is computed as the difference of the revenue supplier gets after and before the change of his location from j to j'. Profit from each of market i from S_1 is

$$(\pi_{ij'} - c_{i\sigma'(i)}) - (\pi_{ij} - c_{i\sigma(i)}) = c_{i\sigma(i)} - c_{i\sigma'(i)}$$

because $\pi_{ij'} = \pi_{ij}$ as the second minimum of the costs for the market i has not changed. Similarly for a market i in S_2 there has been a loss of revenue due to the shift. So the profit is

$$-(\pi_{ij} - c_{i\sigma(i)}) = -((c_{i\sigma'(i)} - c_{i\sigma(i)})$$

because $\pi_{ij} = c_{i\sigma'(i)}$ as the location corresponding to the second minimum among the costs is the new location which will be assigned to the market i. Similar arguments holds for P_3 and P_4

So the total profit

$$P = P_1 + P_2 + P_3 + P_4 = \sum_{i \in M} (c_{i\sigma(i)} - c_{i\sigma'(i)})$$

Also note that the change in potential function

$$\Phi - \Phi' = P$$

So this means that the decrease in the function Φ is exactly equal to the profit of the supplier changing his location. So this is a *Potential Function* by its definition. ■

Corollary 2 *There is a* Deterministic Nash *in this game.*

Proof : As this game is a potential game, a sequence of finite selfish moves leads to a Deterministic Nash. No state is repeated as the potential function Φ strictly decreases after each move leading to a Deterministic Nash. ■

Finally, let us consider the total value of a solution. If the market i is getting supplied by supplier at location $\sigma(i)$, then this contributes $p_i - c_{i\sigma(i)}$ to the total happiness. To see why, let $j = \sigma(i)$, and assume the prize is π_{ij}, then the customer i has benefited $p_i - \pi_{ij}$, while the suppier at j benefited $\pi_{ij} - c_{ij}$ from the sale, so the total benefit is $p_i - c_{i\sigma(i)}$ as claimed. If the market i is not being supplied, than there is no benefit. Using the potential function Φ we defined above, we get that the total benefit of the solution is exactly

$$\sum_i p_i - \Phi.$$

Theorem 3 *The social value of any Deterministic Nash in this game is at least 1/2 of the best possible social value of any solution.*

Proof : Assume that in a Nash solution providers locate at locations $j_1, \ldots j_k$ respectively, and in an optimal solution they locate at locations $j'_1, \ldots, j'_k$. Let $\sigma(i)$ denote the provider assigned to customer i in Nash, and $\sigma'(i)$ the provider assigned in the optimal solution. Now consider the possible move of a provider ℓ moving its location j_ℓ to j'_ℓ. We know from Theorem 1 that the benefit to prover ℓ of this move, is exactly the improvement of the potential function Φ, i.e., exactly the same as the improvement in the total benefit $\sum_i p_i - \Phi$. The solution is a Nash, so the change of moving a single provider does not improve his benefit, and so also does not improve the total benefit. Note, however, that it is possible that moving multiple providers improves the benefit, even if moving a single provider does not. Next we want to bound this improvement possible by the current total benefit of all providers, and this will prove the theorem.

Let $val'(\ell)$ be the new benefit of provider ℓ when he *alone* moves to location j'_ℓ. Consider a custimer i, and let $\delta(i)$ denote the difference between the cost associated with serving customer i in the two solutions. More formally, if i is being served by both solutions then let $\delta(i) = c_{i\sigma(i)} - c_{i\sigma'(i)}$, if i is not served by one of the two solutions, then replace the corresponding cost by p_i in this expression. Now let $I'(\ell)$ be the set of customers served by provider ℓ in the optimal solution. The key observation is that

$$val'(\ell) \geq \sum_{i \in I'(\ell)} \delta(i).$$

This holds with equality if all customers $i \in I'(\ell)$ are served by other providers in the Nash, and the inequality holds, if some customers stay with the same provider.

Now, let $val(\ell)$ be the value of provider ℓ in the Nash solution. This solution is a Nash, so by definition $val'(\ell) \leq val(\ell)$ for all providers ℓ. Adding these inequalities we get the following.

$$\sum_\ell val(\ell) \geq \sum_\ell val'(\ell) \geq \sum_\ell \sum_{i \in I'(\ell)} \delta(i) = \sum_i \delta(i).$$

The right hand side is exactly the improvement in the total improvement in value going from Nash to the optimal solution. Let Φ denote the potential at Nash, and Φ' denote the potential at the optimum. So we get the following bound on the value of the solution.

$$\sum_i p_i - \Phi' = (\sum_i p_i - \Phi) + (\Phi - \Phi') = (\sum_i p_i - \Phi) + \sum_i \delta(i) \leq (\sum_i p_i - \Phi) + \sum_\ell val(\ell) \leq 2(\sum_i p_i - \Phi),$$

where the last inequality follows as $\sum_\ell val(\ell)$ is the providers total value, while $\sum_i p_i - \Phi$ is the total value also including the customers. ■