CS 684 Algorithmic Game Theory	**Scribe: Zoya Svitkina**
Instructor: Eva Tardos	**February 11, 2004**

1 More on nonatomic flows

We will use the following notation as usual:

G is a graph with monotone and continuous delay function $d_e(x)$ on edges. For each $i = 1...k$, there is one unit of flow from s_i to t_i. $\mathcal{P}$ is the set of $s_i - t_i$ paths, and $f_P \geq 0$ for $P \in \mathcal{P}$ is flow on P. $f(e) = \sum_{P:e\in P} f_P$ is the flow (or congestion) on edge e. The delay on a path is $d_P(f) = \sum_{e\in P} d_e(f(e))$.

We know that a flow f is at Nash equilibrium if and only if it minimizes the following objective function:

$$\Phi(f) = \sum_{e\in E} \int_0^{f(e)} d_e(x)dx$$

We would like to say something to the effect that Nash equilibrium is unique, but of course this is not true: in a two-node graph with two parallel edges of constant delay 1, any flow is at Nash equilibrium. But we do have the following.

Theorem 1 *If the delay is monotone non-decreasing and continuous, then, for any edge e, $d_e(f(e))$ is the same for any flow that is at Nash equilibrium.*

Proof. We showed before that Φ is convex. From the theory of convex optimization, we know the following two facts:

1. The set of optimal flows is a convex set; i.e., if f and f' are optimal, then so is $\alpha f + (1-\alpha)f'$, $\alpha \in [0, 1]$.
2. Φ on this convex set is a linear function; i.e., it is of the form $\sum_e (a_e f(e) + b_e)$ for some a and b.

This means that $\int_0^{f(e)} d_e(x)dx$ is linear on the interval between $f(e)$ and $f'(e)$. Then its slope, or derivative, on this interval is constant and is equal to $d_e(f(e)) = d_e(f'(e))$. We conclude that the delay function for edge e is constant for flow values in $(f(e), f'(e))$. ∎

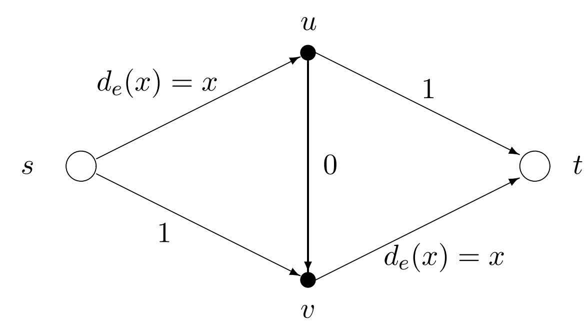
Fact. If f is a Nash flow, then for all i, $P, Q, R \in \mathcal{P}_\rangle$ with $f_P > 0$ and $f_Q > 0$,

$$d_P(f) = d_Q(f) \leq d_R(f)$$

As a result, the notion of "travel time from s_i to t_i" is well-defined, in the sense that it is the same for all paths and in all Nash equilibria.

2 Envy-free flows and Braess paradox

We can define "envy-free" flows as ones in which all paths with positive flow between the same terminals s_i and t_i have the same delay. Then the problem is to make the delay low, subject to the condition that flow be envy-free. At first sight this may seem very similar to Nash equilibrium, but that is not actually the case, as demonstrated in the following example.



If we consider the possible paths from s to v (see figure), it is clear that the selfish players will choose to take the $s-u-v$ path, as its delay is never more than 1. Similarly, to go from u to t, they would choose the path $u-v-t$. As a result, the Nash flow on this graph is for all flow to take the $s-u-v-t$ path, with the delay of 2. The optimal flow on this graph, which happens to be envy-free, is to route half a unit of flow on $s-u-t$, and half on $s-v-t$, for a delay of 1.5.

In a two-node two-edge example that we saw earlier, with delays 1 and x, the optimal flow was also split in half and had a delay of 1.5, but it was not envy-free.

2.1 Opt/Nash ratio

Our objective function here is

$$Opt = C(f) = \sum_P f_P d_P(f)$$

We had two bounds on the Nash to Opt ratio:

1. If for all f, $\Phi(f) \leq C(f) \leq \alpha\Phi(f)$, then $\frac{Nash}{Opt} \leq \alpha$.
2. $max(\frac{Nash}{Opt}) = (max\ \frac{Nash}{Opt}$ on 2-edge, 2-node graph).

Looking at the special case when all delays are linear, i.e., for $a_e \geq 0$ and $b_e \geq 0$, $d_e(x) = a_e x + b_e$, we get the following corollaries corresponding to the two bounds.

1.

$$\begin{aligned} C(f) = \quad & \textstyle\sum_e d_e(f(e))f(e) & = \sum_e (a_e f(e) + b_e) f(e) \\ \Phi(f) = \quad & \textstyle\sum_e \int_0^{f(e)} d_e(x)dx & = \sum_e (\frac{1}{2} a_e f(e)^2 + b_e f(e)) \end{aligned}$$

As we can see, if d is linear, then $max\ \frac{C(f)}{\Phi(f)} \leq 2$.

2. Fact: the worst-case ratio for a two-edge graph and linear delay is 4/3. We will not prove this here, but a useful idea is, instead of thinking of delay as being $ax + b$, to think of it as being *either* a *or* b. The corresponding transformation to the graph is to replace each edge of delay $ax + b$ with two consecutive edges of delay ax and b.