CS 684 Algorithmic Game Theory	**Scribe: Mateo Restrepo**
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Bandwidth sharing on graphs.

Abstract: We extend the results shown in previous lectures on bandwidth sharing on a single link to the case of multiple links. Two pricing mechanisms are discussed for a simplified version of the game in which users have prespecified fixed paths. The first pricing mechanism is the natural extension of Kelly's protocol in which users are price takers. We show that an equilibrium allocation in this mechanism constitutes a social optimum. The second mechanism was proposed recently by Johari and Tsitsiklis. This defines a competitive game between k players, the users that no longer act as price takers. We state without full proof the existence of Nash equilibria for this game as well as $\frac{3}{4}$ bound on the ratio between the social benefit of a Nash equilibrium and that of the social optimum.

1 Bandwidth sharing on graphs — problem formulation.

In the bandwidth sharing problem on graphs we are given a graph $G = (V, E)$ with a bandwidth (capacity) b_e for each edge. There are k users $i = 1, \ldots, k$, each having a source-sink pair s_i-t_i, and a utility function $u_i(x)$ that measures his benefit from a bandwidth x (between s_i and t_i). For the main results that we will develop we will assume that $U_i(x)$ are strictly increasing and strictly concave functions. To keep the mathematics simple we will further assume that these functions are strictly concave and differentiable.

Having said this we present two different versions of the bandwidth sharing problem, the first one simpler than the other.

1.1 Version 1

In this version we are further given fixed paths P_i between s_i-t_i along which each user must route all his traffic. The goal of the problem is to allocate bandwidths $x_1, \ldots, x_k$ to the users in order to maximize the total social welfare without violating capacity constraints on the edges of the graph. The bandwidth sharing problem can be written formally as the following (convex) optimization problem.

$$
\begin{array}{rcl}
\max\limits_{\vec{x}} & & \sum\limits_i U_i(x_i) \\
\text{s. t.} & & \\
x_i & \geq & 0, \forall i \\
\sum\limits_{i: e \in P_i} x_i & \leq & b_e, \forall e \in E
\end{array}
$$

1.2 Version 2

In this version there are no predefined paths for the users and, furthermore, the users are allowed to split their traffic between different paths.

This version of the problem is harder to deal with, and therefore in what follows, we will focus on the first version. However, all results extend to this version as well.

Yet another version, we would not allow users to split traffic between different paths. This version is significantly harder, and the results do not extend to this version.

2 Kelly's game

The problem defined above is a global optimization problem. In what follows we are going to define games that "distribute" the problem among the users and allow them to find optimal or close to optimal solutions based on "local" calculations.

Kelly suggested the following pricing mechanism. Every user offers a certain amount of money w_i^e to each edge e on his path. The edge e then considers all offers and splits bandwidths proportionally. Given the offers of all users to edge e, the bandwidth that i gets is given by

$$x_i^e = \frac{w_i^e}{\sum_j w_j^e} b_e,$$

and is therefore different for each e. The actual bandwidth that user i will be able to use is just $\min_{e \in P_i} x_i^e$ and his benefit is

$$U_i\left(\min_{e \in P_i} x_i^e\right) - \sum_e w_i^e.$$

This game as such is still not well defined. What happens if an edge doesn't receive any monetary offers, that is, if w_j^e? This might happen, for instance, when there is an edge that only appears in one of the users' paths, say in P_1. In this case, no matter how small the offer of player 1 on these edge is, he will get the totality of the bandwidth there. Therefore, the user will always consider that he is wasting the amount of money offered to this edge.

We fix this bug in the mechanism by declaring that on a non-competed ($\sum_j w_j^e = 0$) edge, on which user i claims an amount x_i^e of the bandwidth, we allocate this bandwidth to each of them, only if we have $\sum_j x_j^e \leq b_e$ (otherwise we allocate 0 to all of them). Intuitively, if $\sum_j x_j^e > b_e$ then the edge presents a bottleneck and it better be the case that some users make some offers.

In this setting **Kelly's game** assumes that users are *price takers*. That is, the users don't anticipate the effect of their payment on the prices, but accept the price on every edge as a given fixed quantity p_e. Based on this price each user evaluates the cost of the bandwidth for him and solves the following (one-dimensional) problem to find the bandwidth that gives him the best payoff

$$\max_{x \geq 0} \left(U_i(x) - x \sum_{e \in P_i} p_e \right)$$

Denoting by x_i the optimal bandwidth that solves this problem, the user offers $w_i^e = p_e x_i$ for edge e, and the router sets $x_i^e = x_i$ if $p_e = 0$.

A "Nash" equilibrium[1] in this game is a choice of prices $\vec{p}$ together with allocated bandwidths $\vec{x}$ such that the prices p_e determine bandwidths x_i as just described (the x_i maximize the users' payoffs), while the offers $w_i^e = p_e x_i$ generate prices p_e again, according to the formula $p_e = \frac{1}{b_e}\sum_i w_i^e$.

We state without proof

Lemma 1 *An equilibrium exists for this game (and is unique if the functions are strictly concave)*

Although we don't prove it we give some ideas for the proof. The first idea is to realize that the prices p_e can be seen as dual variables for the inequalities $\sum_{i:e\in P_i} x_i \leq b_e$ and then apply results from duality theory (Lagrange multipliers).

Another approach for the proof would be to use the machinery of Brower's fixed point theorem as it's traditional in game theory.

Easier to prove is the following

Lemma 2 *The resulting allocation at equilibrium is socially optimal.*

Proof. Let $(x_1, \ldots, x_k)$ be the equilibrium allocation and $(x_1^*, \ldots, x_k^*)$ be any other allocation. By definition, the equilibrium allocation satisfies, for every i, $U_i(x_i^*) - x_i^* \sum p_e \leq U_i(x_i) - x_i \sum p_e$, (where the p_e are the prices corresponding to the offers $w_i^e = x_i p_e$ in the equilibrium allocation). Adding up these inequalities over all the users we get

$$\sum_{i=1}^{k}\left(U_i(x_i^*) - x_i^* \sum_{e\in P_i} p_e\right) \leq \sum_{i=1}^{k}\left(U_i(x_i) - x_i \sum_{e\in P_i} p_e\right) \tag{1}$$

For the Nash bandwidths we have in the second term $\sum_i x_i \sum_{e\in P_i} p_e = \sum_e p_e \sum_{i:e\in P_i} x_i = \sum_e p_e b_e$ since, for every e with $p_e \neq 0$ we have $\sum_i x_i = b_e$. On the other hand, it is always the case that $\sum_{i:e\in P_i} x_i^* \leq p_e$ for an arbitrary assignment. This gives us $\sum_{i=1}^{k} x_i^* \sum_{e\in P_i} p_e = \sum_{e\in E}\sum_{i:e\in P_i} x_i^* \leq \sum_e p_e b_e$. Combining the last two results we conclude that

$$\sum_{i=1}^{k} x_i^* \sum_{e\in P_i} p_e \leq \sum_{i=1}^{k} x_i \sum_{e\in P_i} p_e,$$

which together with (1) gives us

$$\sum_{i=1}^{k} U_i(x_i^*) \leq \sum_{i=1}^{k} U_i(x_i)$$

Thus the optimality of the equilibrium assignment. ∎

3 Implementing the Mechanism

Finally, let us comment on implementing Kelly's mechanism. We note that it can be implemented in a simple distributed manner. So the users need to know the total prize $\sum_{e\in P_i} p_e$ to be able to solve the user optimization problem. So the routers on the edges have to somehow send this information along the paths. An edge e need to know the offers w_i^e for all users i on the edge to be able to determine the prize. This is a different number for each edge along P_i, however, the

[1] more properly, "a competitive equilibrium"

users can transmit this information by simply transmitting the chosen rate x_i (where w_i^e is then $x_i p_e$). One can even interpret some of the existing router mechanisms as implementing such prizing games. The idea is that at a reasonable approximation packet loss rate along a paths is the sum of the loss rate along edges. So we can think of the end to end loss rate as encoding the sum of the edge prizes. For the user problem, we can think of different back-off protocols as solving the corresponding user problem with different user utilities. See the paper [3] for more on this.

4 Johari and Tsitsiklis game

As in the previous lecture we go on now to consider the more complicated model that results when we relax the assumption that users are price takers. Johari and Tsitsiklis proposed this game, in which users anticipate the result that their offers will have on the price, as a better approximation of the real problem. Again we are interested in the existence of Nash equilibria for this game, and in the ratio between the optimal social cost and the cost of a Nash equilibrium.

The formal model is as follows. Every user has complete control on his own bids on edges and also realize that the price on each edge is calculated according to $p_e = \frac{1}{b_e}\sum_i w_i^e$. Given the bids $w_{\bar{i}}^e$ of all other users, user i determines his bids w_i^e, on all edges $e \in P_i$, as those maximizing his benefit. Using the notation of [2] (w_i denotes the vector of values w_i^e, $e \in P_i$, $\mathbf{w}_{\bar{i}}$ denotes the set of bids of players other than i, and $Q_i(w_i, \mathbf{w}_{\bar{i}})$ stands for i's benefit, as defined below), the problem solved by each player is

$$\begin{aligned} \max \qquad & Q_i(w_i, \mathbf{w}_{\bar{i}}) \equiv U_i\left(\min_{e\in P_i} \frac{w_i^e}{w_i^e + \sum_{j\neq i} w_j^e} b_e\right) - \sum_{e\in P_i} w_i^e \\ w_i^e \quad & \geq \quad 0, \forall e \in P_i \end{aligned}$$

Notice that this is a multivariate (convex) optimization problem.

The latter defines a game between k players. A Nash equilibrium for this game is a set of bids w_i^e that maximizes the benefit of each player under the assumption that the other players won't change their bids. Formally, w_i^e constitute a Nash equilibrium if for every player i

$$Q_i(w_i, \mathbf{w}_{\bar{i}}) \geq (w_i', \mathbf{w}_{\bar{i}}), \quad \text{for all } w_i' \geq 0$$

Again we have

Lemma 3 *A Nash equilibrium exists.*

We will not prove this result but just comment that the existence follows from the application of standard fixed point procedures to a family of "perturbed" games parameterized by ε that converges in a certain sense to our game. After the a limiting procedure a sequence of Nash equilibria for these games converges to a Nash equilibrium for the present game. For more details we refer the interested reader to Theorem 6. p. 20 of [2].

Finally, we have the following result about the price of anarchy in this situation.

Theorem 4 *Let x_i be a Nash allocation of bandwidths to users. Then the social benefit derived from is at least $\frac{3}{4}$ the maximum possible social benefit. In symbols*

$$\sum_i U_i(x_i) \geq \frac{3}{4} \max_{x_i^*} \sum_i U_i(x_i^*),$$

were the maximum on the right hand side is taking over all assignments that satisfy bandwidth limits.

Proof sketch. We need to prove that there is a Nash equilibrium. For now we concentrate on comparing an equilibrium to the optimal solution.

In a Nash equilibrium, the bandwidth that a user gets in different links across his path must be the same. For if there were any edge giving a greater bandwidth to this user, then the user could lower his bid, still retaining the value of $U(x_i)$ but increasing his benefit. From this we conclude that for the purposes of finding a Nash equilibrium, the maximization problem solved by each user is not actually a multivariate problem. There is only one variable that captures all what is essential to user i and that is the (uniform) bandwidth x_i assigned to his path.

Now consider the optimization faced by user i, given that all other users bids are fixed. Each edge has an amount W^e offered to the edge by other users. In order to get bandwidth x this user has to offer payment w_i^e such that $x/b^e = w_i^e/(W^e + w_i^e)$, so user will pay $w_i^e = W^e x/(b_e - x)$ for an edge e along his path. Now his overall utility will be

$$U_i(x) - \sum_{e \in P_i} w_i^e = U_i(x) - \sum_{e \in P_i} W^e x/(b_e - x).$$

By our assumption that the utility functions are convex, this gets maximized when we set the derivative to 0. So in a Nash equilibrium, we must have that user i's bandwidth x_i satisfies.

$$U_i'(x_i) - \sum_{e \in P_i} (W^e/(b_e - x_i) + W^e x/(b_e - x_i)^2) = 0.$$

As we have done in the case of a single edge, we simplify this expression by substituting, the edges unit price: $p_e = (W^e + w_i^e)/b_e = W^e/(b_e - x_i)$ to get:

$$0 = U_i'(x_i) - \sum_{e \in P_i} (p_e + p_e x_i/(b_e - x_i)) = U_i'(x_i) - \sum_{e \in P_i} \frac{p_e}{1 - x_i/b_e}.$$

Recall from the previous lecture that, in the case of a single link of capacity B, an assignment $(x_1, \ldots, x_k)$ is in equilibrium if, for each i, $U_i'(x_i)\left(1 - \frac{x_i}{B}\right) = p$ where $p = \frac{B}{\sum w_i}$ was the price per unit of capacity on the link. Our calculation above shows that for the present case we get analogously that the assignment of bandwidths is in equilibrium if and only if, for each user i,

$$U_i'(x_i) = \sum_{e \in P_i} \frac{p_e}{1 - x_i/b_e} \tag{2}$$

In last expression, the role of each one of the edges is nicely separated in each of the terms of the sum. This now allows us to proceed in more or less the same way as in the case of a single link and obtain the desired result ⊓

5 Complementary References

Most of this lecture was based on the first reference which extended the results of the second reference.

1. F. P. Kelly, *Charging and rate control for elastic traffic,* European Transactions on Telecommunications, vol. 8, pp. 33-37, 1997.

2. Ramesh Johari, and John N. Tsitsiklis. (2004). *Efficiency loss in a network resource allocation game.*

3. S. H. Low, Larry Peterson, and Limin Wang: Understanding Vegas: A Duality Model, Journal of the ACM, 49 (2) 2007-235, 2002.