CS 684 Algorithmic Game Theory	**Scribe: Chandrashekhar Nagarajan**
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1 Randomized rounding and truthful auction

We consider the same combinatorial auction setup as in last class. There are k "single minded bidders" $1, 2, 3 \dots k$ and each bidder i wants a subset S_i of items from a collection of items N: $S_i \subseteq N$. Each bidder i has an utility u_i associated with his set S_i. The bidder pays p_i toward getting the set S_i. His happiness is $u_i - p_i$ if he gets the set or $-p_i$ if he does not get the set.

Note that bidders need not reveal their exact utility. So we saw that the VCG mechanism of pricing gives a social optimal solution. But given the true S_i and u_i values

$$max \sum_{i \in I} u_i \ s.t\ sets\ corresponding\ to\ players\ in\ I\ are\ disjoint$$

is NP-complete and also no good approximation algorithm exists.

But this problem is easy to approximate if we have the sets of high multiplicity. Let the multiplicity of an item j be m_j and there are are totally n different items. Let us assume the following :

- $m_j \geq c \cdot log(n)$ and all the copies are indistinguishable.
- Each user wants atmost one copy of an item.

This problem is also NP-Hard but it is easy to approximate this problem and good approximations exists. One of the algorithm for this is *randomized rounding.*

Let us have $\{0, 1\}$ variables x_i for a bidder i. $x_i = 1$ if i gets the set S_i and $x_i = 0$ otherwise. Consider the LP relaxation of the optimization problem.

$$Max \sum_i u_i \cdot x_i$$

$$\sum_{i: j \in S_i} x_i \leq m_j \quad \forall\ distinct\ items\ j$$

$$0 \leq x_i \leq 1 \quad \forall\ bidders\ i$$

The last constraint is the LP relaxation of the integer programming problem as we are allowing the integer variable x_i to take fractional values.

Now the main idea of randomized rounding is as follows: Give the set S_i to the bidder i independently with probability x_i. This makes sense as $0 \leq x_i \leq 1\ \forall i$.

Note that the expected sum of utilities $= E(\sum_{i \in I} u_i) = \sum_i u_i \cdot x_i$ since expectation is linear. Note that $\sum_i u_i \cdot x_i \geq$ Social Optimum as the integer optimum is also a solution to the LP and we are maximizing over fractional solutions also. Also note that

$$E(\#\ copies\ of\ an\ item\ j\ sold) = \sum_{i: j \in S_i} x_i \leq m_j$$

To keep the # of each item sold well under control we decrease the probability of giving the set S_i to the bidder i from x_i to $(x_i/1+\delta)$.

So now

$$E(\sum_{i \in I} u_i) = \sum_i \frac{u_i \cdot x_i}{1+\delta}$$

$$E(\#\ of\ copies\ of\ item\ j\ sold) = \sum_{i:j \in S_i} \frac{x_i}{1+\delta} \leq \frac{m_j}{1+\delta}$$

If $Y_1, Y_2, \ldots Y_n$ are independent $\{0,1\}$ variables with $E(\sum Y_i) \leq m$ and $\delta > 0$ then by Chernoff Bounds

$$Prob(\sum Y_i > (1+\delta)m) \leq \left(\frac{e}{(1+\delta)^{(1+\delta)}}\right)^m$$

Applying this result here,

$$Prob(\sum_{i:j \in S_i} x_i > m_j) \leq \left(\frac{e}{(1+\delta)^{(1+\delta)}}\right)^{\frac{m_j}{1+\delta}}$$

We can select c and δ such that this probability is less than ϵ/n.

Corollary 1 *Probability that there is an over sold item is less than or equal to ϵ.*

This is because there are atmost n items and probability that each item is oversold is $\leq \epsilon/n$. ■

So if there is an oversold item during the auction, cancel the auction and rerun it once again. So with probability ϵ the auction gets canceled. So we can use this procedure to implement truthful auction.

Now we shift our attention to bidders/users. Users are risk neutral and they want to maximize their expected happiness. Let user i gets his set S_i with probability y_i. The user pays the price p_i and his expected profit $= u_i y_i - p_i$. So every user tend to maximize $u_i y_i - p_i$.

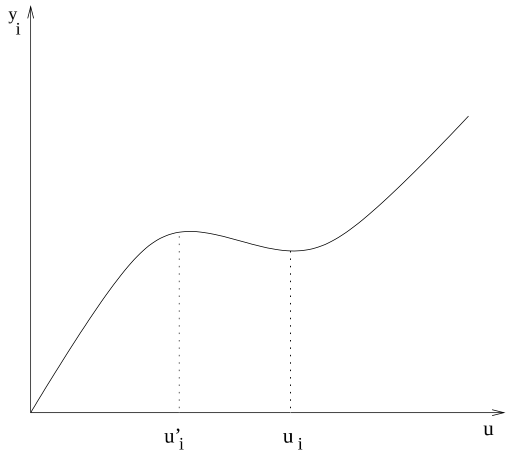

Figure 1: Non-monotone probability function

Let us see what happens if the probability function is not monotone increasing on the utility stated u_i. Consider the case in Figure 1.

If a player i's original utility is u_i and if the probability of getting his set increases when he lies about his utility to $u_i' < u_i$ then the player can increase his chance of getting the set by lying about his utility. We claim that this makes the auction untruthful. To see this consider the expected price p_i and p_i' that the user has to pay in the two cases above. If $p_i' \leq p_i$, then the user would increase his change of getting the item and decrease his cost by claiming utility u_i' when the true utility is u_i, so the auction is not truthful. So we must have $p_i' > p_i$. Now assume the user's true utility is u_i', then the expected benefit of user i claiming utilities u_i and u_i' respectively is $u_i'y_i - p_i$, and $u_i'y_i' - p_i'$. For the auction being truthful, we must have that $u_i'y_i - p_i \leq u_i'y_i' - p_i'$, and hence $p_i - p_i' \geq u_i'(y_i' - y_i) > 0$, a contradiction.

Theorem 2 *If y_i is a monotone increasing probability function of utility of player i which denotes the probability of the player i getting his set and the price he pays is $p_i = u_i y_i(u_i) - \int_0^{u_i} y_i \, du_i$ $\forall i$ then the players will behave truthfully.*

Proof:

A player i has expected profit $= \int_0^{u_i} y_i \, du_i$ if he says his utility truthfully.

Case 1: Let a player i lie about his utility as $u_i' > u_i$.

So the probability of i getting the set S_i is $y_i(u_i')$.

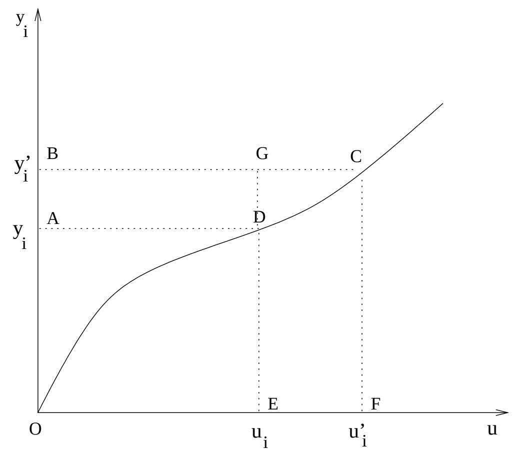

Figure 2: Case 1: $u_i' > u_i$

$$
\begin{aligned}
Expected\ profit &= u_i y_i(u_i') - (u_i' y_i(u_i') - \int_0^{u_i'} y_i \, du_i) \\
&= \int_0^{u_i'} y_i \, du_i - (u_i' - u_i) y_i(u_i') \\
&= \int_0^{u_i} y_i \, du_i - Area(GDC) \quad (from\ Figure\ 2) \\
&\leq \int_0^{u_i} y_i \, du_i \quad (As\ y_i\ is\ monotone\ increasing)
\end{aligned}
$$

So the bidder has no incentive to lie by increasing his utility.

Case 2: If the player i lies about his utility as $u_i' < u_i$.

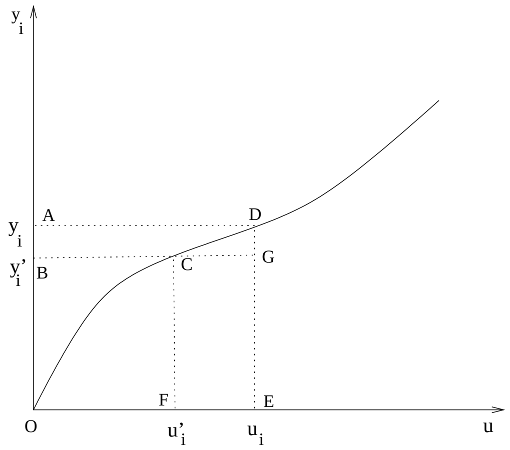

Figure 3: Case 2: $u_i' < u_i$

$$\begin{aligned}
Expected\ profit &= u_i y_i(u_i') - (u_i' y_i(u_i') - \int_0^{u_i'} y_i\ du_i) \\
&= \int_0^{u_i'} y_i\ du_i + (u_i - u_i') y_i(u_i') \\
&= \int_0^{u_i} y_i\ du_i - Area(CGD) \quad (from\ Figure\ 3) \\
&\leq \int_0^{u_i} y_i\ du_i \quad (As\ y_i\ is\ monotone\ increasing)
\end{aligned}$$

So the player i has no incentive to lie as his expected profit only goes down if he lies. So this setting is truthful if y_i is monotone increasing with u_i. ■

Note that x_i which is a result of solving the LP relaxation is monotone increasing with the utility stated u_i, and so $(x_i/1+\delta)$ is also monotone increasing. But actually there is a slight problem as by decreasing u_i a player can affect the probability of over sold item and so can affect the probability of cancelling the auction which may be an incentive to lie. To get around this difficulty the auctioneer can cancel the auction with a fixed probability ϵ. This makes the auction truthful.