$\boxed{\text{9 Dec 2024}}$ Sampling Matchings

Recall. Glauber dynamics samples proper colorings of a graph with n vertices, max degree Δ, by repeatedly selecting random vertex, random color in $[q]$, recoloring vtx if it doesn't match any neighbor's color.

Upshot of Fri lecture. If $n(t)$ denotes Hamming dist of 2 colorings X_t, X'_t, then

$$\mathbb{E}\left[n(t+1) \mid n(t)\right] \leq n(t) - \left(1 - \frac{4\Delta}{q}\right)\frac{n(t)}{n}$$

$$= \left[1 - \left(1 - \frac{4\Delta}{q}\right)\frac{1}{n}\right] \cdot n(t),$$

If $q > 4\Delta$, then $1 - \frac{4\Delta}{q} \geq \frac{1}{q}$

$$\mathbb{E}\left[n(t+1) \mid n(t)\right] \leq \left(1 - \frac{1}{qn}\right) \cdot n(t)$$

$$\mathbb{E}\left(n(t+1)\right) \leq \left(1 - \frac{1}{qn}\right) \mathbb{E}\left[n(t)\right]$$

$$\mathbb{E}\left[n(t)\right] \leq \left(1 - \frac{1}{qn}\right)^t \cdot n(0) \leq \left(1 - \frac{1}{qn}\right)^t \cdot n$$

To make $\Pr(n(t) > 0) < \epsilon$,

choose $t \geq qn \ln(n/\varepsilon)$. ← upper bound on $\tau_{mix}(\epsilon)$.

Then $\left(1 - \frac{1}{qn}\right)^t \leq \left(1 - \frac{1}{qn}\right)^{qn \ln(n/\varepsilon)}$

$$< \left(\frac{1}{e}\right)^{\ln(n/\varepsilon)} = \frac{\varepsilon}{n}.$$

We've seen if π_0 = init state distrib.

π_t = marginal time t distrib.

then

$$\pi_t^T = \pi_0^T P^t$$

So if π denotes stationary distrib, then estimating mixing times boils down to asking how fast $\pi_0^T P^t \longrightarrow \pi^T$ in $\|\cdot\|_1$.

Lemma. If eigenvalues of P (reversible Markov trans mtx) are $1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$ then

$$\tau_{mix}(\epsilon) \leq \frac{1}{1-\lambda_2} \cdot \ln\left(\frac{1}{\varepsilon \cdot \pi_{min}}\right)$$

where $\pi_{min} = \min\{\pi(x) \mid \pi(x) > 0\}$.

(π denotes stationary distrib.)

Proof. Application of Cauchy-Schwarz.

Proof. $\pi_x P_{xy} = \pi_y P_{yx} \Longrightarrow \frac{P_{xy}}{\pi_y} = \frac{P_{yx}}{\pi_x}$.

If ν is any vector, and $\frac{\nu}{\pi}$ denotes the vector $\left(\frac{\nu}{\pi}\right)_x = \frac{\nu_x}{\pi_x}$, then

$$\left[P\cdot\left(\frac{\nu}{\pi}\right)\right]_x = \sum_y \frac{P_{xy}\nu_y}{\pi_y} = \sum_y \frac{P_{yx}\nu_y}{\pi_x} = \frac{1}{\pi_x}\left(P^T\nu\right)_x$$

$$\therefore \quad P\cdot\left(\frac{\nu}{\pi}\right) = \frac{P^T\nu}{\pi}.$$

If $\pi_0, \pi_1, \pi_2, \ldots$ are the marg. distribs. of the state at time $0, 1, 2, \ldots$

$$\pi_t = \left(P^T\right)^t \pi_0$$

$$\frac{\pi_t}{\pi} = \frac{\left(P^T\right)^t \pi_0}{\pi} = P^t\left(\frac{\pi_0}{\pi}\right).$$

Recall $P\cdot 1 = 1$.

$$1 - \frac{\pi_t}{\pi} = P^t\left(1 - \frac{\pi_0}{\pi}\right).$$

$$\|\pi_t - \pi\|_{TV} = \frac{1}{2}\sum_x \left|\pi_{t,x} - \pi_x\right|$$

$$= \frac{1}{2}\sum_x \pi_x \left|\frac{\pi_{t,x}}{\pi_x} - 1\right|$$

$$\leq \frac{1}{2}\left(\sum_x \pi_x\left(\frac{\pi_{t,x}}{\pi_x} - 1\right)^2\right)^{1/2} \underbrace{\left(\sum_x \pi_x\right)^{1/2}}_{=1}$$

$$= \frac{1}{2}\left\langle \frac{\pi_t}{\pi} - 1, \frac{\pi_t}{\pi} - 1\right\rangle_\pi^{1/2}$$

$$= \frac{1}{2}\left\| P^t\left(\frac{\pi_0}{\pi} - 1\right)\right\|_\pi$$

$$\le \frac{1}{2} \lambda_2^t \left\| \frac{\pi_0}{\pi} - 1 \right\|_\pi$$

$$\le \frac{1}{2} \lambda_2^t \frac{1}{\pi_{min}}$$

$T_{mix}(\varepsilon)$ in the lemma ensures RHS $\le \varepsilon$.

Now suppose we have Markov chain with N states $(N = 2^{O(n)})$ transition matrix P, reversible w.r.t. π.

We want to be able to show

$$\frac{1}{1-\lambda_2} \le \text{poly}(n).$$

An observation: Let $\langle x, y \rangle_\pi = \sum_{i \in \Omega} \pi_i x_i y_i$.

Observe, by reversibility,

$$\langle x, Py \rangle_\pi = \sum_{i \in \Omega} \pi_i x_i \left(\sum_j P_{ij} y_j \right)$$

$$= \sum_{i \in \Omega} \sum_{j \in \Omega} x_i \left(\pi_j P_{ji} y_j \right)$$

$$= \sum_{j \in \Omega} \pi_j \left(\sum_{i \in \Omega} P_{ji} x_i \right) y_j$$

$$= \langle Px, y \rangle_\pi$$

P is self-adjoint w.r.t. π.

$\therefore$ $\mathbb{1}_N - P$ is self-adjoint w.r.t. π.

In fact $\mathbb{1}_N - P$ is the normalized Laplacian of a graph with $V = \Omega$ edge weights $w(i,j) = \pi_i P_{ij}$.

Cheeger $\Rightarrow$ $(1 - \lambda_2) \geq \frac{1}{8}(\text{min sparsity})^2$.

Can bound min sparsity away from $\emptyset$ by finding a concurrent MCF of rate $\frac{1}{\text{poly}(n)}$.

"Canonical paths method"
(Jerrum & Sinclair)
employs this method to analyze a MC for sampling bipartite matchings.