$\boxed{\text{6 Dec 2024}}$ Mixing Times and Couplings

<u>Recap.</u> Metropolis algorithm takes:

(i) state set Ω

(ii) unnormalized weight func $w : \Omega \to \mathbb{R}_{\geq 0}$, $w \neq 0$

(iii) "proposal distribution" K: symmetric Markov transition matrix on Ω.

Metropolis Procedure:

In state $i \in \Omega$ at time t $(X_t = i)$

(1) Sample random $j \in \Omega$ with probability K_{ij}.
"propose transition (i,j)"

(2) with probability $\min\left(1, \frac{w(j)}{w(i)}\right)$ accept transition:
$$X_{t+1} = j$$

(3) else $X_{t+1} = i$.

In equations this means Metropolis Markov chain has transition matrix

$$P_{ij} = \begin{cases} K_{ij} \cdot \min\left(1, w(j)/w(i)\right) & \text{if } i \neq j \\ 1 - \sum_{k \neq i} P_{ik} & \text{if } i = j \end{cases}$$

<u>Lemma.</u> Let $\pi = \frac{w}{Z}$ where $Z = \sum_{i \in \Omega} w(i)$.

P is reversible w.r.t. π. $\therefore$
π is a stationary distribution of P.

Proof. Must show $\forall i,j \quad \pi_i P_{ij} = \pi_j P_{ji}$

If $i=j$, trivial. So assume $i \neq j$.

$$\pi_i P_{ij} = \frac{w(i)}{Z} \cdot K_{ij} \cdot \min\left(1, \frac{w(j)}{w(i)}\right)$$

$$= \frac{K_{ij}}{Z} \cdot \min(w(i), w(j))$$

$$\pi_j P_{ji} = \frac{K_{ji}}{Z} \cdot \min(w(j), w(i))$$

The RHS are equal.

Example. For $\Omega = [q]^{V(G)}$ where $V(G)$ is the vertex set of a graph G, $[q] = \{1, \ldots, q\}$ is a set of colors,

$$w(x) = \begin{cases} 1 & \text{if } x(u) \neq x(v) \ \forall \text{ edge } \{u,v\} \\ \emptyset & \text{otherwise} \end{cases}$$

Let $n = |V(G)|$.
Suppose proposal matrix K is

$$K_{ij} = \begin{cases} \frac{1}{qn} & \text{if } \exists v \in V(G) \text{ s.t. } j(u) = i(u) \ \forall u \neq v, \\ 0 & \text{otherwise} \end{cases}$$

Metropolis transition matrix:

$$P_{ij} = \begin{cases} \frac{1}{qn} & \text{if } \exists v \in V(G) \text{ s.t. } j(u) = i(u) \ \forall u \neq v \text{ and } j \text{ is a proper coloring} \\ 0 & \text{otherwise} \end{cases}$$

"Glauber dynamics"

We will analyze with $q > 4 \cdot \Delta$ — max degree of G.

<u>Def:</u> The ε-mixing time of a Markov chain, $\tau_{min}(\varepsilon)$, is the smallest $t_0 \in \mathbb{N}$ such that for every initial ~~state~~ distribution π and every $t \geq t_0$,

$$\| \pi^T P^t - \pi^* \|_{TV} \leq \varepsilon$$

where $\pi^* =$ the stationary distribution.

We aim to prove: for Glauber with $q > 4\Delta$

$\tau_{mix}(\varepsilon) \leq \text{poly}(n, q, \log(1/\varepsilon))$.

<u>Proof by coupling:</u> Construct a joint distribution on <u>pairs of state sequences</u> $\begin{bmatrix} (X_0, X_1, X_2, \ldots) \\ (X_0', X_1', \ldots\ldots) \end{bmatrix}$

such that:

(i) X_0 drawn from π, X_0' drawn from π^*

(ii) $X_0, X_1, X_2, \ldots$ and $X_0', X_1', X_2', \ldots$ both have the Markov transition dynamics given by P.

(iii) $\Pr(X_t \neq X_t') \leq \varepsilon$ for all $t \geq t_0$.

Coupling for Glauber dynamics.

Propose the same transition (i_t, j_t') in both chains at all t.

How to analyze?

Let $n(t) = \#\{v \mid X_t(v) \neq X_t'(v)\}$.

$n(t)$ decreases to $n(t)-1$ when the proposed transition picks some vertex v where $X_t(v) \neq X_t'(v)$ and recolors with color c in both chains

$$\Pr\left(n(t+1) = n(t)-1\right) \geq \underbrace{\frac{n(t)}{n}}_{X_t(v) \neq X_t'(v)} \cdot \underbrace{\frac{q-2\Delta}{q}}_{c \text{ acceptable in both chains.}}$$

$n(t)$ increases to $n(t)+1$ when the proposed vertex v has a neighbor w in one chain (not the other) with the proposed color, c.

$$\Pr\left(n(t+1) = n(t)+1\right) \leq \underbrace{\frac{n(t)}{n} \cdot \Delta}_{v \text{ is a neighbor of a } w \text{ s.t. } X_t(w) \neq X_t'(w)} \cdot \underbrace{\frac{2}{q}}_{\Pr \text{ that } c \text{ is in } \{X_t(w), X_t'(w)\}}$$