2 Dec 2024

<u>Recap.</u> Semidefinite program (SDP)

$$\max \quad \frac{1}{4}\mathrm{Tr}(L_G A)$$

$$\text{s.t.} \quad A \text{ symm, PSD } (A \succeq 0)$$

$$A_{uu} = 1 \quad \forall u \in V$$

A solution, A, can be efficiently factorized as $A = X^T X$ for some matrix X whose columns $\{x_u \mid u \in V\}$ satisfy

$$A_{uv} = \langle x_u, x_v \rangle \quad \forall u,v.$$

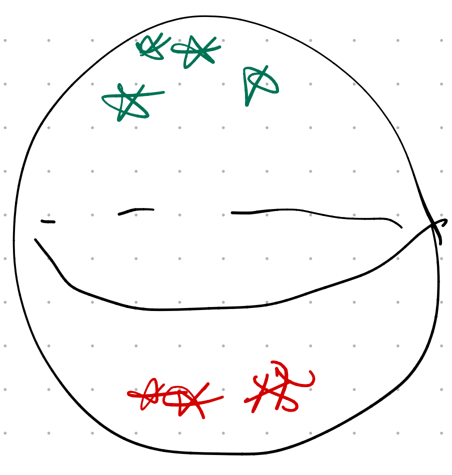
Objective function is

$$\frac{1}{4}\mathrm{Tr}(L_G A) = \frac{1}{4}\sum_{\{u,v\}\in E} w(u,v)\left(A_{uu} - A_{uv} - A_{vu} + A_{vv}\right)$$

$$= \frac{1}{4}\sum_{\{u,v\}\in E} w(u,v)\left[2 - 2\langle x_u, x_v\rangle\right]$$

$$= \frac{1}{2}\sum_{\{u,v\}\in E} w(u,v)\left[1 - \langle x_u, x_v\rangle\right].$$

$$= \frac{1}{2}\sum_{\{u,v\}\in E} w(u,v)\left(1 - \cos(\theta_{uv})\right).$$



Sample a uniform random unit vector $z \in \mathbb{R}^n$

Partition V into $A = \{u \mid \langle z, x_u \rangle \geq 0\}$

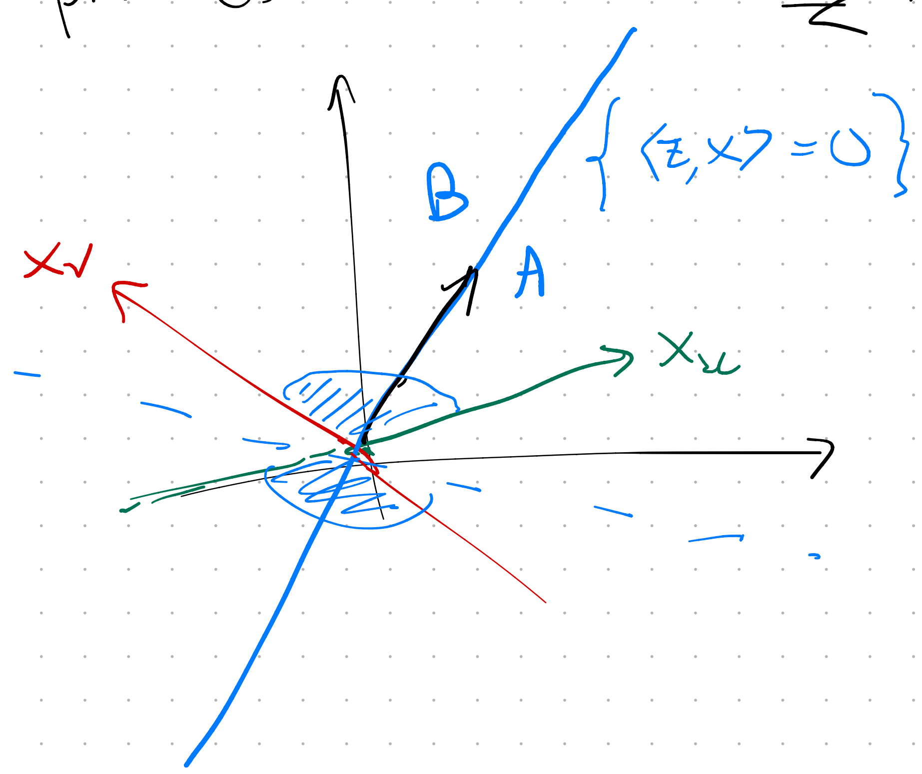
$B = \{u \mid \langle z, x_u \rangle < 0\}$.

Want to compare $\mathbb{E}[w(\partial A)]$ with

$$\frac{1}{2} \sum_{\{u,v\} \in E} w(u,v)(1 - \cos\theta_{uv}).$$

$$\mathbb{E}[w(\partial A)] = \sum_{\{u,v\} \in E} w(u,v) \cdot \Pr(|\{u,v\} \cap A| = 1)$$

In the 2D subspace spanned by x_u, x_v, vectors are partitioned into 2 sets according to sign of their inner product with z.



$$\Pr(\{u,v\} \text{ is cut}) = \frac{\theta_{uv}}{\pi}$$

So the approx factor obtained by Goemans Williamson is $\geq \inf_{\theta \in [0, 2\pi)} \frac{\theta/\pi}{\frac{1}{2}(1 - \cos\theta)} \approx 0.878$

Sampling from Implicitly Specified Distributions (Unnormalized)

Given a function $w : \Omega \to \mathbb{R}_{\geq 0}$ where $|\Omega| = 2^{O(n)}$ and w can be computed in poly(n) time.

Problem. Draw a random sample from Ω with prob. $p(\omega) \propto w(\omega)$.

$$p(\omega) = \frac{w(\omega)}{Z} \quad \text{where } Z = \sum_{\omega \in \Omega} w(\omega).$$