$\boxed{\text{25 Nov 2024}}$ Max Cut and SDP

Obs. If $G \overset{=}{} (V,E)$ is undir graph with $w(u,v) = \begin{cases} 1 & \text{if } \{u,v\} \in E \\ 0 & \text{o.w.} \end{cases}$
and $x \in \{\pm 1\}^V$ then

$$\langle x, L_G x \rangle = \sum_{\{u,v\} \in E} (x_u - x_v)^2$$

$$= 4 \cdot \mathrm{Cut}(x)$$

where $\mathrm{Cut}(x)$ = # of edges labeled with $\{+1, -1\}$ by x.

So $\mathrm{MaxCut}(G) = \max \left\{ \langle x, L_G x \rangle \;\middle|\; x \in \{\pm 1\}^V \right\}$.

So.... apply random thresholding to the eigenvector x corresponding to $\lambda_{max}(L_G)$?

E.g. $G = K_n$ clique on n vertices.

$$L_G = \begin{bmatrix} n-1 & -1 & -1 & \cdots & -1 \\ -1 & n-1 & & & \vdots \\ -1 & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ -1 & \cdots & & & n-1 \end{bmatrix} = n \cdot \mathrm{Id}_n - 11^T$$

Make $w(u,v) = \begin{cases} 2 & \text{if } u = v_1 \text{ or } v = v_1 \\ 1 & \text{otherwise} \end{cases}$

For this G

$$L_G = \begin{bmatrix} 2(n-1) & -2 & -2 & \cdots & -2 \\ -2 & n & -1 & \cdots & -1 \\ \vdots & -1 & n & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ -2 & -1 & \cdots & \cdots & n \end{bmatrix} = (n+1) \cdot Id_n - 11^T + ye_1^T + e_1 y^T$$

where $y^T = \left[\frac{n-2}{2} \quad -1 \quad -1 \quad \cdots \quad -1 \right]$.

Now the λ_{max} eigenvector is nearly parallel to e_1 and very misaligned with genuine max-cut vectors.

<u>Observation.</u> $\langle x, L_G x \rangle = x^T L_G x$

$$= \mathrm{Tr}(L_G x x^T).$$

So $\max \left\{ \langle x, L_G x \rangle \;\middle|\; \langle x, x \rangle = n \right\}$

is the same as:

$$\begin{aligned} \max \quad & \mathrm{Tr}(L_G X) \\ \text{s.t.} \quad & X \in \mathbb{R}^{n \times n}, \quad X = X^T \\ & \mathrm{Tr}(X) = n \\ & X \succeq 0 \quad (\text{i.e. } X \text{ is PSD}) \\ & \mathrm{rank}(X) = 1 \end{aligned}$$

Instead try to solve:

$$\text{(GW-SDP)} \quad \left.\begin{array}{ll} \max & \mathrm{Tr}(L_G \cdot X) \\ \text{s.t.} & X \in \mathbb{R}^{n \times n}, \text{ symmetric, PSD} \\ & \mathrm{Tr}(X) = n \quad \text{[redundant]} \\ & x_{ii} = 1 \quad \forall i \in [n] \\ & \cancel{\mathrm{rank}(X) = 1} \end{array}\right\} \text{semidef. program.}$$

SDP: Maximize/minimize a linear function of the entries of a square matrix subj to:

- the matrix is symmetric PSD
- any number of additional linear inequality constraints on the matrix entries.

Solve GW-SDP.

Let $X = W^T W$.

Let $\{w_1, \ldots, w_n\}$ be columns of W.

$$X_{ij} = \langle w_i, w_j \rangle.$$

Sample unit vector $u \in \mathbb{R}^n$
uniformly at random.

$$S = \{ i \mid \langle w_i, u \rangle > 0 \}$$

$$V - S = \{ i \mid \langle w_i, u \rangle \leq 0 \}.$$

Why does this work?