22 Nov 2024

Cheeger's Inequality, pt. 2

Recap. $\lambda_2(\bar{L}_G) = \min \left\{ \frac{\sum_{\{u,v\}\in E} w(u,v)(y_u - y_v)^2}{\frac{1}{2d(V)} \sum_u \sum_v d(u)d(v)(y_u - y_v)^2} \;\middle|\; y \not\parallel \vec{1} \right\}$

$$\frac{1}{2}\phi(S) \leq h(S) \leq \phi(S)$$

$$\lambda_2(\bar{L}_G) \leq \min_{S \neq \emptyset, V} \{\phi(S)\}$$

To show: $\min_{S \neq \emptyset, V} \{\phi(S)\} \leq \sqrt{8\lambda_2(\bar{L}_G)}$.

Suppose $y \in \mathbb{R}^V$, $y \not\parallel \vec{1}$, let

$$Q(y) = \frac{\sum w(u,v)(y_u - y_v)^2}{\frac{1}{2d(V)} \sum\sum d(u)d(u)(y_u - y_v)^2}.$$

We will show $\exists$ threshold θ such that $S = \{v \mid y_v < \theta\}$ satisfies $h(S) \leq \sqrt{2Q(y)}$.

$$\phi(S) \leq 2h(S) \leq \sqrt{8Q(y)}.$$

Transformations. Add a multiple of $\vec{1}$ if necessary so that

(1) $$\sum_{v:\, y_v < 0} d(v) \leq \frac{1}{2} d(V)$$

(2) $$\sum_{v:\, y_v > 0} d(v) \leq \frac{1}{2} d(V).$$

From now on assume $\vec{y}$ satisfies (1), (2).

Let $$z_u = \begin{cases} y_u^2 & \text{if } y_u \geq 0 \\ -y_u^2 & \text{if } y_u < 0. \end{cases}$$

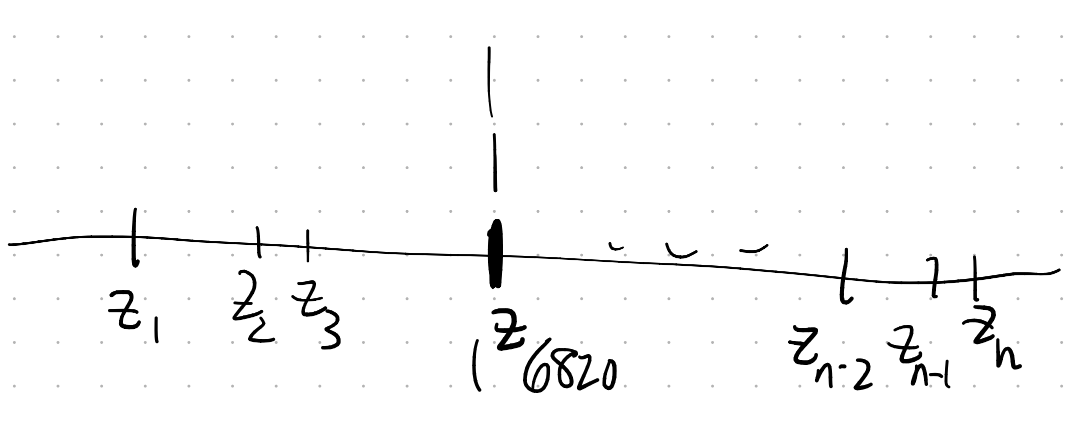
Lastly assume $\underbrace{\max\{z_u\}}_{z_{max}} - \underbrace{\min\{z_u\}}_{z_{min}} = 1.$

If $z_{max} - z_{min} = \alpha \neq 1$, change y to $\frac{1}{\alpha} \cdot y$.

Now choose $\theta \in [z_{min}, z_{max}]$ unif. random. Let

$$S = \{ v \mid z_v < \theta \}.$$

Work on estimating $E[w(\partial S)]$ and $E[\min\{d(S), d(V-S)\}]$.



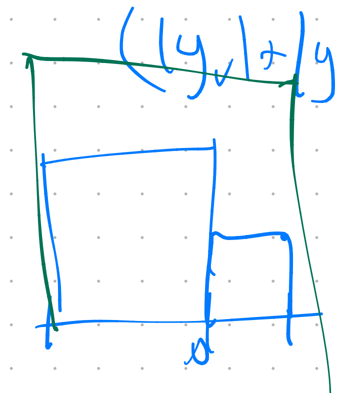
$$\mathbb{E}\left[\min\{d(S), d(V-S)\}\right] = \sum_u \mathbb{E}\left[d(u)\cdot \mathbb{1}\quad\right]$$

$$= \sum_u d(u)\cdot \Pr\left(u \in \arg\min\{d(S), d(V-S)\}\right),$$

$$= \sum_u d(u)\cdot |z_u| = \sum_u d(u)\, y_u^2 = \langle y, y\rangle_D$$

$$\mathbb{E}\left[w(\partial S)\right] = \sum_{\substack{\{u,v\}\in E\\ z_u \le z_v}} \Pr(z_u < \theta \le z_v)\cdot w(u,v)$$

$z_v - z_u \le (y_v - y_u)\cdot(|y_v| + |y_u|)$.



$$= \sum_{z_u \le z_v} w(u,v)\,(z_v - z_u)$$

$$\le \sum_{z_u \le z_v} w(u,v)(y_v - y_u)(|y_v| + |y_u|)$$

$$\le \left[\sum_{z_u \le z_v} w(u,v)(y_v - y_u)^2\right]^{\frac{1}{2}} \left[\sum_{z_u \le z_v} w(u,v)(|y_u| + |y_v|)^2\right]^{\frac{1}{2}}$$

$(a+b)^2 \le 2a^2 + 2b^2$

$$\le \langle y, L_G y\rangle_D^{\frac{1}{2}} \left[2 \sum_{z_u \le z_v} w(u,v)(y_u^2 + y_v^2)\right]^{1/2}$$

$$= \langle y, L_G y\rangle_D^{\frac{1}{2}} \left[\sum_u \sum_v w(u,v)(y_u^2 + y_v^2)\right]^{1/2}$$

$$= \langle y, \bar{L}_G y\rangle_D^{\frac{1}{2}} \left[\sum_u \sum_v w(u,v)\, y_u^2 + \sum_u \sum_v w(u,v)\, y_v^2 \right]^{\frac{1}{2}}$$

$$= \langle y, \bar{L}_G y\rangle_D^{\frac{1}{2}} \left[\sum_u d(u)\, y_u^2 + \sum_v d(v)\, y_v^2 \right]^{1/2}$$

$$= \langle y, \bar{L}_G y\rangle_D^{1/2} \cdot \left(2 \langle y, y\rangle_D \right)^{1/2}$$

$$\frac{\mathbb{E}\{ w(\partial S) \}}{\mathbb{E}\{ \min\{ d(S), d(V-S)\}\}} \leq \frac{\sqrt{2}\, \langle y, \bar{L}_G y\rangle_D^{1/2} \langle y, y\rangle_D^{1/2}}{\langle y, y\rangle_D}$$

$$= \sqrt{2 R(y)}$$