20 Nov 2024 Cheeger's Inequality

Recap. G undirected graph, (symmetric) edge weights $w(u,v) \geq 0$.

$d(v) = \sum_u w(u,v)$ "(weighted) degree of v"

Assume $d(v) > 0$ for all v.

Standard inner product $\langle x, y \rangle = \sum_v x_v y_v$

Degree-weighted inner product $\langle x, y \rangle_D = \sum_v d(v) x_v y_v$

Symmetric matrices L_G (Laplacian), D_G (diagonal degree matrix)

Normalized Laplacian $\overline{L}_G = D_G^{-1} L_G$.

$$\langle x, \overline{L}_G y \rangle_D = \langle x, L_G y \rangle = \langle L_G x, y \rangle = \langle \overline{L}_G x, y \rangle_D$$

$$\langle x, L_G x \rangle = \sum_{\{u,v\} \in E} w(u,v) (x_u - x_v)^2$$

Eigenvalues $0 = \lambda_1(\overline{L}_G) \leq \lambda_2(\overline{L}_G) \leq \cdots \leq \lambda_n(\overline{L}_G)$.

$$\lambda_2(\overline{L}_G) = \min \left\{ \langle y, \overline{L}_G y \rangle_D \;\middle|\; \langle y, y \rangle_D = 1, \; \langle y, \mathbb{1} \rangle_D = 0 \right\}$$

$$= \min \left\{ \frac{\langle y, \overline{L}_G y \rangle_D}{\langle y, y \rangle_D} \;\middle|\; y \neq 0, \; \langle y, \mathbb{1} \rangle_D = 0 \right\}$$

For $y \in \mathbb{R}^V$ let $\overline{y} \in \mathbb{R}^V$ be the (unique) vector parallel to $\vec{\mathbb{1}}$ such that

$\langle y - \overline{y}, \mathbb{1} \rangle_D = 0$.

$$\overline{y} = \frac{\langle y, \mathbb{1} \rangle_D}{\langle \mathbb{1}, \mathbb{1} \rangle_D} \cdot \vec{\mathbb{1}}$$

$$\{ y \neq 0, \langle y, 1 \rangle_D = 0 \} = \{ y' - \bar{y}' \mid y' - \bar{y}' \neq 0 \}$$

$$\lambda_2(\bar{L}_G) = \min \left\{ \frac{\langle y - \bar{y}, \bar{L}_G (y - \bar{y}) \rangle_D}{\langle y - \bar{y}, y - \bar{y} \rangle_D} \;\middle|\; y - \bar{y} \neq 0 \right\}$$

Obs. 1 $\langle y - \bar{y}, \bar{L}_G (y - \bar{y}) \rangle_D = \langle y, \bar{L}_G y \rangle_D$

$$= \sum_{\{u,v\} \in E} w(u,v) (y_u - y_v)^2$$

Obs. 2 Turn V into a prob space by assigning $\Pr(u) = \frac{d(u)}{d(V)}$.

Let Y be a random variable taking the value y_u at vertex u.

Then $\mathbb{E}[Y] = \frac{1}{d(V)} \sum_u d(u) y_u$

$$= \frac{1}{d(V)} \langle y, 1 \rangle_D = \frac{\langle y, 1 \rangle_D}{\langle 1, 1 \rangle_D}.$$

If $\overline{Y}$ denotes the rand var. taking value $\overline{y}_u$ at vertex u,

$\overline{Y}$ = constant function $\mathbb{E}[Y]$.

$$\mathrm{Var}(Y) = \mathbb{E}\left[(Y-\overline{Y})^2\right]$$

$$= \frac{1}{d(V)} \sum_{u \in V} d(u) \left(y_u - \overline{y}_u\right)^2$$

$$= \frac{1}{d(V)} \langle y-\overline{y}, y-\overline{y} \rangle_D .$$

On a sample space $V \times V$ with $\Pr((u,v)) = \dfrac{d(u) \cdot d(v)}{d(V)^2}$.

we have indep random vars

$Y_1(u,v) = y_u \quad Y_2(u,v) = y_v$.

Y_1 and Y_2 are indep each distributed as Y.

$$\mathrm{Var}(Y_1 - Y_2) = \mathrm{Var}(Y_1) + \mathrm{Var}(-Y_2)$$
$$= 2 \cdot \mathrm{Var}(Y)$$
$$= \frac{2}{d(V)} \cdot \langle y - \bar{y}, y - \bar{y} \rangle_D$$

$$\mathbb{E}(Y_1 - Y_2) = 0,$$

$$\mathbb{E}\left[(Y_1 - Y_2)^2 \right] = \frac{2}{d(V)} \langle y - \bar{y}, y - \bar{y} \rangle_D$$
$$\parallel$$
$$\sum_{u \in V} \sum_{v \in V} \frac{d(u)\, d(v)}{d(V)^2} (y_u - y_v)^2$$

$$\langle y - \bar{y}, y - \bar{y} \rangle_D = \frac{1}{2 \cdot d(V)} \sum_u \sum_v d(u) d(v) (y_u - y_v)^2$$

Lemma

$$\lambda_2(\bar{L}_G) = \min \left\{ \frac{\langle y, L_G y \rangle}{\frac{1}{2d(V)} \sum_u \sum_v d(u) d(v) (y_u - y_v)^2} \;\middle|\; y \not\parallel \mathbf{1} \right\}$$

For a vertex ~~set~~ S consider $y^S \in \{0,1\}^V$ defined by

$$y^S_u = \begin{cases} 1 & \text{if } u \in S \\ 0 & \text{if } u \notin S. \end{cases}$$

If $S \neq \emptyset, V$ then $y^S \not\parallel \mathbf{1}$.

For $y = y^S \not\parallel \mathbf{1}$,

$$\langle y, L_G y \rangle = \sum_{\{u,v\} \in E} w(u,v)(y_u - y_v)^2$$

$$= w(\partial S).$$

$$\frac{1}{2d(V)} \sum_{(u,v)\in V^2} d(u)d(v)(y_u - y_v)^2$$

$$= \frac{1}{2d(V)} \left[\underbrace{\boxed{\sum_{u\in S} \sum_{v \notin S} d(u)d(v)}}_{= \left(\sum_{u\in S} d(u)\right)\cdot\left(\sum_{v\notin S} d(v)\right)} + \sum_{u\notin S} \sum_{v\in S} d(u)d(v) \right]$$

$$= \frac{\cancel{2}\, d(S)\, d(V-S)}{\cancel{2}\, d(V)}$$

$$\frac{\langle y^S, L_G y^S \rangle}{\frac{1}{2d(V)} \sum_u \sum_v d(u)d(v)(y^S_u - y^S_v)^2}$$

$$= \frac{w(\partial S) \cdot d(V)}{d(S)\, d(V-S)} = \phi(S),$$

$$\lambda_2(\bar{L}_G) \le \min_{S \ne \emptyset, V} \{\phi(S)\},$$