18 Nov 2024

Using the Laplacian for lower bounds on sparsest cut

Recap. G an edge weighted graph with symmetric weights $w(u,v) = w(v,u) > 0$ if $\{u,v\} \in E(G)$, $w(u,v) = 0$ if $u = v$ or $\{u,v\} \notin E$.

$$h(S) = \frac{w(\partial S)}{\min\{d(S), d(V-S)\}} \qquad \phi(S) = \frac{w(\partial S) \cdot d(V)}{d(S) \cdot d(V-S)}$$

$$\tfrac{1}{2}\phi \le h \le \phi$$

$$(L_G)_{uv} = \begin{cases} d(u) & \text{if } u = v \\ -w(u,v) & \text{if } u \neq v \end{cases}$$

$$x^T L_G x = \sum_{\{u,v\} \in E} (x_u - x_v)^2 \cdot w(u,v)$$

The smallest eigenvalue of L_G is always $\emptyset$.

The multiplicity of this eigval is > 1 if and only if G is disconnected.

This lecture: if 2nd smallest normalized Laplacian eigenvalue is far from zero then $\phi(S)$ is far from zero for all $S \neq \emptyset$ s.t. $V-S \neq \emptyset$.

Def. The normalized Laplacian of G is

$$\overline{L_G} = D_G^{-1} L_G.$$

I_G has diagonal entries $= 1$

row sums $= 0$

column sums typically $\neq 0$

<u>Algebra review:</u> An inner product on a vector space V is a bilinear function $V \times V \to \mathbb{R}$, that is symmetric:

$$\langle v, w \rangle = \langle w, v \rangle.$$

We say $\langle \cdot, \cdot \rangle$ is positive definite if

$\langle v, v \rangle \geq 0 \quad \forall v$, with equality only when $v = 0$.

<u>Example.</u> The dot product $\langle v, w \rangle = \sum_{i=1}^{n} v_i w_i$.

The degree-weighted inner product

$$\langle x, y \rangle_D = \sum_{u \in V(G)} d(u) \cdot x_u y_u$$

$$= \langle x, D_G y \rangle$$

<u>Def.</u> If V is a vector space,

$\langle , \rangle_V$ is an inner product on V

$T : V \to V$ linear transformation

("endomorphism")

We say T is self-adjoint w.r.t. $\langle , \rangle_V$

if $\forall x, y \in V \quad \langle x, Ty \rangle_V = \langle Tx, y \rangle_V$.

Remark: T is self-adj w.r.t. standard inner product iff the matrix representing T is symmetric.

Facts. If a matrix A represents a finite dimensional self-adjoint endomorphism of V w.r.t. a pos. def. inner product $\langle \cdot, \cdot \rangle_V$ then:

(i) All eigenvalues of A are real.

(ii) V has a basis of eigenvectors $v_1, \ldots, v_n$ of A that are mutually orthogonal w.r.t. $\langle \cdot, \cdot \rangle_V$:

$$\langle v_i, v_j \rangle_V = 0 \text{ for } i \neq j.$$

(iii) If $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ are the eigenvalues corresponding to $v_1, \ldots, v_n$,

$$\lambda_1 = \min_{\langle x,x \rangle_V = 1} \{ \langle x, Ax \rangle_V \}$$

$$\lambda_n = \max_{\langle x,x \rangle_V = 1} \{ \langle x, Ax \rangle_V \}$$

(iv) [Courant - Fischer] More generally,

$$\lambda_k = \min_{\substack{\dim W = k \\ W \subseteq V}} \max_{\substack{\langle x,x \rangle_V = 1 \\ x \in W}} \{ \langle x, Ax \rangle_V \}.$$

$$\lambda_{n-k+1} = \max_{\dim W = k} \min_{x \in W, \langle x,x \rangle_V = 1} \{ \langle x, Ax \rangle_V \}$$

E.g., $\langle \cdot, \cdot \rangle_V$ = standard inner product on $\mathbb{R}^n$.

$$A = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

$$W = \begin{bmatrix} \text{any} \\ \text{any} \\ \vdots \\ \text{any} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} a \\ a \\ a \\ a \end{matrix}}\right\} k \\ \left.\vphantom{\begin{matrix} a \\ a \\ a \end{matrix}}\right\} n-k \end{matrix}$$

$$\mathcal{U} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \text{any} \\ \text{any} \\ \vdots \\ \text{any} \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} a \\ a \\ a \\ a \end{matrix}}\right\} k-1 \\ \left.\vphantom{\begin{matrix} a \\ a \\ a \\ a \end{matrix}}\right\} n-k+1 \end{matrix}$$

If $\tilde{W} \subset \mathbb{R}^n$ has $\dim(\tilde{W}) = k$ then

$\dim(\tilde{W} \cap \mathcal{U}) > 0$.

So $\exists\, x \in \tilde{W} \cap \mathcal{U}$, $\langle x, x \rangle = 1$.

$$\langle x, Ax \rangle = \sum_{i=k}^{n} \lambda_i x_i^2,$$

$$\geq \lambda_k \sum_{i=k}^{n} x_i^2 = \lambda_k.$$

(v) $$\lambda_k = \min_{\substack{x \in V \\ \langle x, v_i \rangle_V = 0 \\ \text{for } 1 \leq i < k \\ \langle x, x \rangle = 1}} \left\{ \langle x, Ax \rangle_V \right\}$$

v_k is the vector that attains this minimum.

Fact. $\overline{L}_G$ is self-adjoint w.r.t. $\langle \, , \rangle_D$.

Why? We must show

$$\langle x, \overline{L}_G y \rangle_D = \langle \overline{L}_G x, y \rangle_D$$

$$\langle x, D_G^{-1} L_G y \rangle_D = \langle D_G^{-1} L_G x, y \rangle_D$$

$$\langle x, \cancel{D_G D_G^{-1}} L_G y \rangle = \langle D_G^{-1} L_G x, D_G y \rangle$$

$$= \langle \cancel{D_G D_G^{-1}} L_G x, y \rangle$$

$$\langle x, L_G y \rangle = \langle L_G x, y \rangle$$

What is $\lambda_1(\overline{L}_G)$?

$$\langle x, \overline{L}_G x \rangle_D = \langle x, L_G x \rangle$$

The minimum is $\emptyset$, attained when x is parallel to $\vec{1}$.

$$\therefore \quad \lambda_1 = 0$$

$$v_1 = [d(V)]^{-1/2} \cdot \vec{1}$$

What is $\lambda_2(\overline{L}_G)$?

$$\lambda_2(\overline{L}_G) = \min\left\{ \langle x, \overline{L}_G x \rangle_D \;\middle|\; \begin{array}{l} \langle x, x \rangle_D = 1 \\ \langle x, \vec{1} \rangle_D = 0 \end{array} \right\}$$