$\boxed{\text{15 Nov 2024}}$ The Graph Laplacian

Recall: For an edge set C, with respect to src-sink pairs $\{(s_i, t_i)\}_{i=1}^{k}$

$$\text{Sparsity}(C) = \frac{\text{cap}(C)}{\text{sep}(C)}.$$

"Sparsest edge cut": find edge set C minimizing this quantity.

For a vertex set S let

$$\partial S = \{ \text{edges } \{u,v\} \mid u \in S, \ v \notin S \}$$

"sparsest vertex cut": find vtx set S minimizing

$$h(S) = \frac{\text{cap}(\partial S)}{\min\{d(S), d(V \setminus S)\}}$$

where $d(S) := \sum_{v \in S} \text{degree}(v)$.

How to reduce sparse vertex cut to sparse edge cut?

Def. $\phi(S) := \frac{\text{cap}(\partial S)}{d(S) \cdot d(V \setminus S)} \cdot d(V).$

Relation between $h(S)$ and $\phi(S)$...

$$\frac{1}{2}\phi(S) \leq h(S) \leq \phi(S)$$

$$= \frac{cap(S)}{\min(d(S), d(V-S))} \cdot \frac{d(V)}{\max(d(S), d(V-S))}$$

$$d(S) + d(V-S) = d(V)$$

$$\frac{1}{2} d(V) \leq \max\{d(S), d(V-S)\} \leq d(V)$$

Let $k = d(V)^2$.

For each ordered vertex pair (u,v) we create $d(u) \cdot d(v)$ commodities with $s_i = u$, $t_i = v$.

What is $sep(\partial S)$?

$$sep(\partial S) = \#\{i \mid s_i \in S, t_i \in V-S\} + \#\{i \mid s_i \in V-S, t_i \in S\}.$$

$$= \sum_{u \in S} \sum_{v \in V-S} \#\{i \mid s_i = u, t_i = v\} + \#\{i \mid s_i = v, t_i = u\}$$

$$= \sum_{u \in S} \sum_{v \in V-S} 2\, d(u)\, d(v)$$

$$= 2\, d(S)\, d(V-S).$$

what is $d(V)^2 = 4|E|^2 = 4m^2$

$$d(V) = \sum_{v \in V} \text{degree}(v) = 2 \cdot |E|$$

The Laplacian of a Graph

If G is a graph (undirected) with edge capacities $w(u,v) = w(v,u) \geq 0$ (no edge from u to v represented by $w(u,v) = 0$) (always $w(u,u) = 0$.)

let $d(v) := \sum_{u \neq v} w(u,v)$

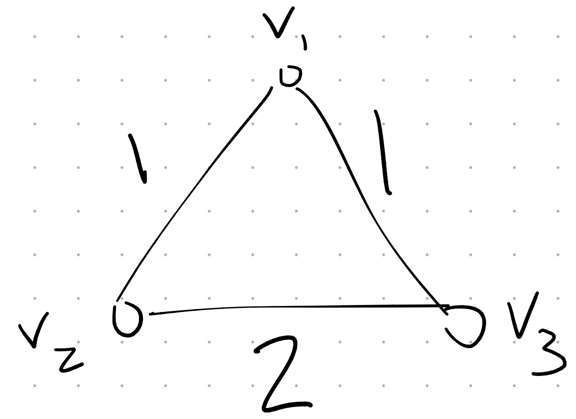
The matrix D_G ("degree matrix") is

$$(D_G)_{uv} = \begin{cases} d(v) & \text{if } u = v \\ 0 & \text{if } u \neq v \end{cases}$$

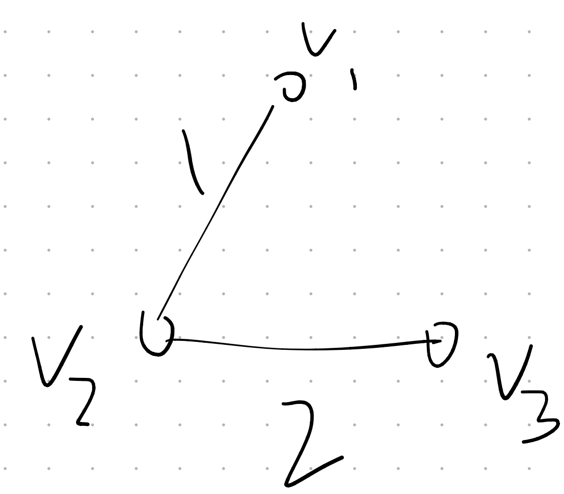
The Laplacian L_G is

$$(L_G)_{uv} = \begin{cases} d(v) & \text{if } u = v \\ -w(u,v) & \text{if } u \neq v. \end{cases}$$

Example.



$$D_G = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad L_G = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 3 & -2 \\ -1 & -2 & 3 \end{bmatrix}$$



$$D_G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad L_G = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 2 \end{bmatrix}$$

Facts. D_G, L_G are symmetric.

$$L_G \cdot \vec{1} = \vec{0}.$$

D_G, L_G are positive semi definite!

RECALL. For an $n \times n$ symmetric matrix A the following are equivalent

("A is positive semidefinite.")

1. $\forall v \in \mathbb{R}^n \quad v^T A v \geq 0$

2. $\exists B \in \mathbb{R}^{n \times n}$ s.t. $A = BB^T$

3. $\exists$ vectors $v_1, \ldots, v_n$ s.t.

$$A_{ij} = v_i^T v_j \qquad \forall i, j$$

(A is the "Gram matrix" of $v_1, \ldots, v_n$.)

4. All eigenvalues of A are ≥ 0.

For $x \in \mathbb{R}^n$,

$$x^T L_G x = \sum_{u \in V} \sum_{v \in V} x_u (L_G)_{uv} x_v$$

$$= \sum_{u \in V} x_u^2 \cdot d(u)$$

$$+ \sum_{u \in V} \sum_{v \neq u} x_u \left(-w(u,v)\right) x_v$$

$$= \sum_{u \in V} \sum_{v \neq u} x_u^2 \cdot w(u,v)$$

$$- \sum_{u \in V} \sum_{v \neq u} x_u x_v \, w(u,v)$$

$$= \sum_{\{u,v\} \in E(G)} w(u,v) \cdot \left[x_u^2 + x_v^2 - 2x_u x_v \right]$$

$$= \sum_{\{u,v\} \in E} w(u,v) \left(x_u - x_v \right)^2$$

$$\geq 0$$

$$\lambda_{min}(L_G) = 0$$

because $L_G \cdot \vec{1} = \vec{0}$.

When is the multiplicity of eigenvalue $\emptyset$ greater than 1?

$\dim(\text{nullspc}(L_G))$

$= \#$ conn compts of G