13 Nov 2024 — Using Hedge to Solve Multicommodity Flow.

Recap: $G=(V,E)$ directed

Capacity $c(e)=1$ $\forall e$.

$Q=(P_1,\dots,P_k)$ ranges over $\prod_{i=1}^{k} P(s_i,t_i)$ ← paths from s_i to t_i

$n_Q(e) :=$ # paths in Q that contain e.

Max concurrent flow rate

$$r^* = \max \sum_Q y_Q$$

$$\text{s.t.} \quad \sum_Q n_Q(e)\, y_Q \le 1 \quad \forall e$$

$$y_Q \ge 0 \quad \forall Q$$

Equivalently:

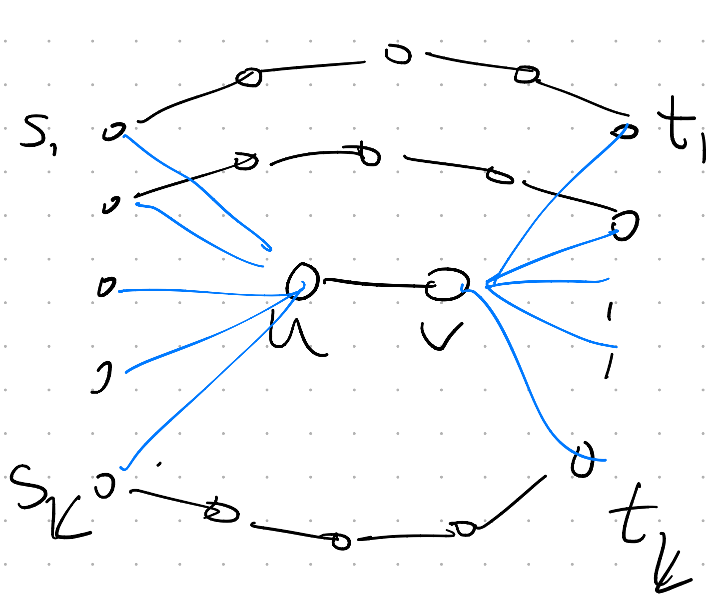
$$\frac{1}{r^*} = \min \ K$$

$$\text{s.t.} \quad \sum_Q \hat{y}_Q = 1$$

$$\sum_Q n_Q(e)\, \hat{y}_Q \le K \quad \forall e$$

$$\hat{y}_Q \ge 0 \quad \forall Q$$

Greedy fails:



Let $\text{MaxHedge}_\varepsilon$ be the following algorithm that choses probabilities $p_t(i)$ $(i \in [n])$ in response to a sequence of vectors $g_1, g_2, \ldots, g_{t-1} \in [0,1]^n$

Set $G_{t-1} = g_1 + \cdots + g_{t-1}$

$$w_t(i) = (1+\varepsilon)^{G_{t-1}(i)} \quad \text{for } i \in [n]$$

$$W_t = \sum_{i=1}^{n} w_{t-1}(i)$$

$$p_t(i) = \frac{w_t(i)}{W_t}.$$

Theorem. $\displaystyle\sum_{t=1}^{T} \langle g_t, p_t \rangle \geq (1-\varepsilon)\left[\max_{i \in [n]} \sum_{t=1}^{T} g_t(i)\right] - \frac{\ln n}{\varepsilon}.$

Multicommodity Flow Algorithm

for $t = 1, \ldots, T$:

let $\{p_t(e)\}_{e \in E} = \text{MaxHedge}_\varepsilon(g_1, \ldots, g_{t-1})$

for $i \in [k]$ let P_i = min cost $s_i t_i$ path using costs $p_t(e)$.

let $Q_t = (P_1, \ldots, P_k)$.

for $e \in E$ let $g_t(e) = \dfrac{n_{Q_t}(e)}{k}$

output $\hat{y}$ defined by $\hat{y}_Q = \frac{1}{T} \#\{t : Q_t = Q\}$

For this algorithm we get some

$$\hat{K} = \max_{e \in E} \left\{ \sum_Q n_Q(e) \hat{y}_Q \right\}$$

and we hope $\hat{K}$ not much bigger

than $K^* = \frac{1}{r^*} = \min \left\{ \max_e \left(\sum_Q n_Q(e) y_Q \right) \middle| \begin{array}{l} \sum_Q y_Q = 1 \\ y_Q \geq 0 \end{array} \right\}$

Analysis. We aim to show that if T is large enough $\hat{K} \leq \frac{1}{1-2\varepsilon} \cdot K^*$.

Means $\frac{1}{\hat{K}} \geq (1-2\varepsilon) \frac{1}{K^*} = (1-2\varepsilon) r^*$.

And $y = \frac{\hat{y}}{\hat{K}}$ is a feasible MCF of rate $\frac{1}{\hat{K}}$.

Important observation. There exists a MCF y^* satisfying $\sum_Q y^*_Q = 1$,

$$\forall e \quad \sum_Q n_Q(e) y^*_Q \leq K^*.$$

For any prob distrib $p(e)$ on edges,

$$\sum_{e,Q} p(e)\, n_Q(e)\, y_Q^* \leq K^*$$

$$\sum_Q y_Q^* \left(\sum_e p(e)\, n_Q(e) \right) \leq K^*$$

So if Q is selected to minimize $\sum_e p(e)\, n_Q(e)$, we know for sure

$$\sum_e p(e)\, n_Q(e) \leq K^*.$$

In particular $\forall t \in [T]$,

$$\sum_e p_t(e)\, n_{Q_t}(e) \leq K^*$$

$$\sum_e p_t(e) \frac{n_{Q_t}(e)}{k} \leq \frac{K^*}{k}$$

$$\forall t \in [T] \quad \langle g_t, p_t \rangle \leq \frac{K^*}{k}$$

$$T\frac{K^*}{k} \geq \sum_{t=1}^{T} \langle g_t, p_t \rangle$$

$$\frac{\ln m}{\varepsilon}$$

$$\parallel$$

$$\geq (1-\varepsilon) \max_{e \in E} \left(\sum_{t=1}^{T} g_t(e) \right) - \frac{\ln |E|}{\varepsilon}$$

$$= \frac{1-\varepsilon}{k} \max_{e \in E} \left(\sum_{t=1}^{T} n_{Q_t}(e) \right) - \frac{\ln m}{\varepsilon}$$

$$K^* \geq (1-\varepsilon) \max_{e \in E} \left(\sum_{Q} \hat{y}_Q n_Q(e) \right) - \frac{k \ln m}{\varepsilon T}$$

$$= (1-\varepsilon) \hat{K} - \frac{k \ln m}{\varepsilon T}$$

Want: $K^{*} \geq (1-2\varepsilon)\hat{K}$

Then we need:

$$\frac{k \ln m}{\varepsilon T} \leq \varepsilon \cdot \hat{K}$$

$$T \geq \frac{k \ln m}{\varepsilon^2 \hat{K}}$$

$\hat{K} \geq \frac{k}{m}$ no ~~matter~~ what.

So $T = \frac{k \ln m}{\varepsilon^2 \cdot k/m} = \frac{m \ln m}{\varepsilon^2}$

iterations suffice.