8 Nov 2024 Game Playing Using Multiplicative Weights

Announcements: Remaining workload in .6820.

i. Prob Set 4 — to be released tonight
Due in 2 weeks.

ii. Final project — see upcoming Ed post.
Due during exam week.

A two-player zero sum game is given by two (finite) strategy sets

$[n]$ (Player 1)
$[m]$ (Player 2)

and a "payoff matrix" U.

If P1 chooses $i \in [n]$, P2 chooses $j \in [m]$

P1's payoff is u_{ij}
P2's payoff is $-u_{ij}$

Each player aims to maximize their payoff.

E.g. Rock-Paper-Scissors

$n = m = 3$

u_{ij}	$j = 1$	2	3
$i = 1$	0	-1	1
2	1	0	-1
3	-1	1	0

A mixed strategy is a vector of probabilities

$$p \in \Delta([n]) \quad \text{or} \quad q \in \Delta([m]).$$

$$\begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix} \quad p_i \geq 0 \ \forall i \quad \sum_{i=1}^{n} p_i = 1 \qquad \begin{bmatrix} q_1 \\ \vdots \\ q_m \end{bmatrix} \quad q_j \geq 0 \ \forall j \quad \sum_{j=1}^{m} q_j = 1$$

The expected payoffs for P1 and P2 are

$$\sum_{i,j} p_i q_j u_{ij} \quad \text{and} \quad \sum_{i,j} p_i q_j (-u_{ij})$$

$$p^T U q \qquad\qquad -p^T U q.$$

We say p is a best response to q, $p \in BR(q)$, if $p^T U q = \max_{\tilde{p} \in \Delta(n)} \{\tilde{p}^T U q\}$.

Similarly $q \in BR(p)$ if $p^T U q = \min_{\tilde{q} \in \Delta(m)} \{p^T U \tilde{q}\}$

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p and q are an equilibrium if $p \in BR(q)$ and $q \in BR(p)$.

von Neumann's <u>Minimax Theorem</u> For all 2PZSGs with finite strategy sets,

$$\min_q \max_p p^T U q = \max_p \min_q p^T U q.$$

Furthermore if p^*, q^* are such that

$$\min_q \{ (p^*)^T U q \} = \max_p \{ p^T U q^* \}$$

then (p^*, q^*) is a mixed equilibrium.

Should be "obvious" that

$$\min_q \max_p p^T U q \geq \max_p \min_q p^T U q$$

To prove LHS $\geq$ RHS it is necessary and sufficient to show

$$\forall \tilde{q} \quad \max_p p^T U \tilde{q} \geq \max_p \min_q p^T U q$$

That holds b/c $\forall p \quad p^T U \tilde{q} \geq \min_q p^T U q.$

For the reverse inequality

$$\min_q \max_p p^T U q \leq \max_p \min_q p^T U q$$

we present an algorithmic construction.

<u>Observation 1.</u> vN minimax relation invariant under adding a constant to all entries of U.

WLOG $U \geq 0$.

<u>Observation 2</u>: Also invariant to scaling.

WLOG $0 \leq u_{ij} \leq 1 \quad \forall i, j$.

Adopt the perspective of column player, repeatedly playing the game against a row player.

In round t if row player chooses $i(t)$, p_t
column player experiences loss $\ell_{i(t),j} = u_{i(t),j}$
if they chose $j \in [m]$.

Organize these losses into a sequence of vectors $\ell_1 = \ell_{i(1)}, \ \ell_2 = \ell_{i(2)}, \ \ldots$

$\ell_t = U^T p_t$.

$$\begin{bmatrix} | & | & \\ \ell_1 & \ell_2 & \cdots \\ | & | & \end{bmatrix}$$

Column player using HEDGE_ε algorithm would choose sequence of mixed strategies $q_1, q_2, \ldots, q_T \in \Delta[m]$, Row player chooses $p_t \in BR(q_t)$.

Let $\bar{q} = \frac{1}{T}\left(q_1 + q_2 + \dots + q_T\right)$

$$\sum_t \bar{q} U^T p_t = \frac{1}{T} \sum q_t^T U^T p_t$$

$$= \frac{1}{T} \sum \langle l_t, q_t \rangle$$

$$\leq \frac{1}{T}\left[(1+\varepsilon) \min_q \sum_{t=1}^{T} \langle l_t, q \rangle + \frac{\ln m}{\varepsilon}\right]$$