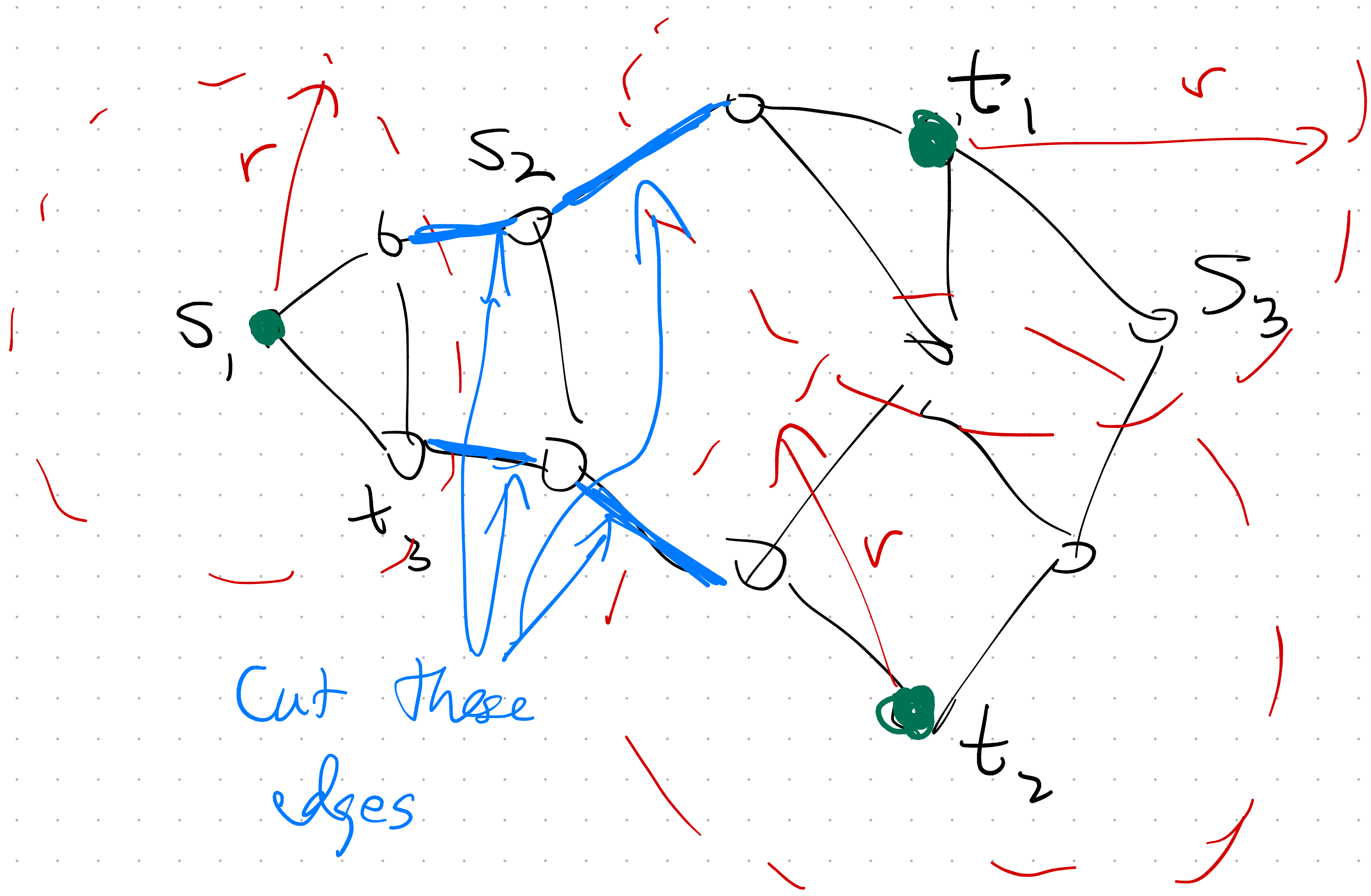
$\boxed{\text{1 Nov 2024}}$ Finishing approx max flow min cut.



Fractional cut: $X = (x_e)$ defining shortest-path distances dist_x such that

$$\sum_{i=1}^{k} \text{dist}_x(s_i, t_i) = 1.$$

Volume of x: $\displaystyle\sum_e c(e) x_e$

<u>Goal</u>: Postprocess x to find an edge cut set C (randomly sampled) s.t.

$$\mathbb{E}[\text{cap}(C)] \leq \text{volume of } x$$

$$\mathbb{E}(\text{sep}(C)) \geq \frac{1}{O(\log k)}$$

$$\Pr(C \text{ separates } s_i \text{ from } t_i)$$

$$= \int_0^1 \Pr(\mathrm{dist}_x(s_i, W) < r < \mathrm{dist}_x(t_i, W))\, dr$$

$$+ \int_0^1 \Pr(\mathrm{dist}_x(t_i, W) < r < \mathrm{dist}_x(s_i, W))\, dr$$

$$\geq \int_0^{\frac{1}{2}\mathrm{dist}_x(s_i,t_i)} \underbrace{\langle \text{sum of same 2 probabilites} \rangle}\, dr$$

To show: $\forall r$ this integrand is $\geq \frac{\text{const}}{\log k}$

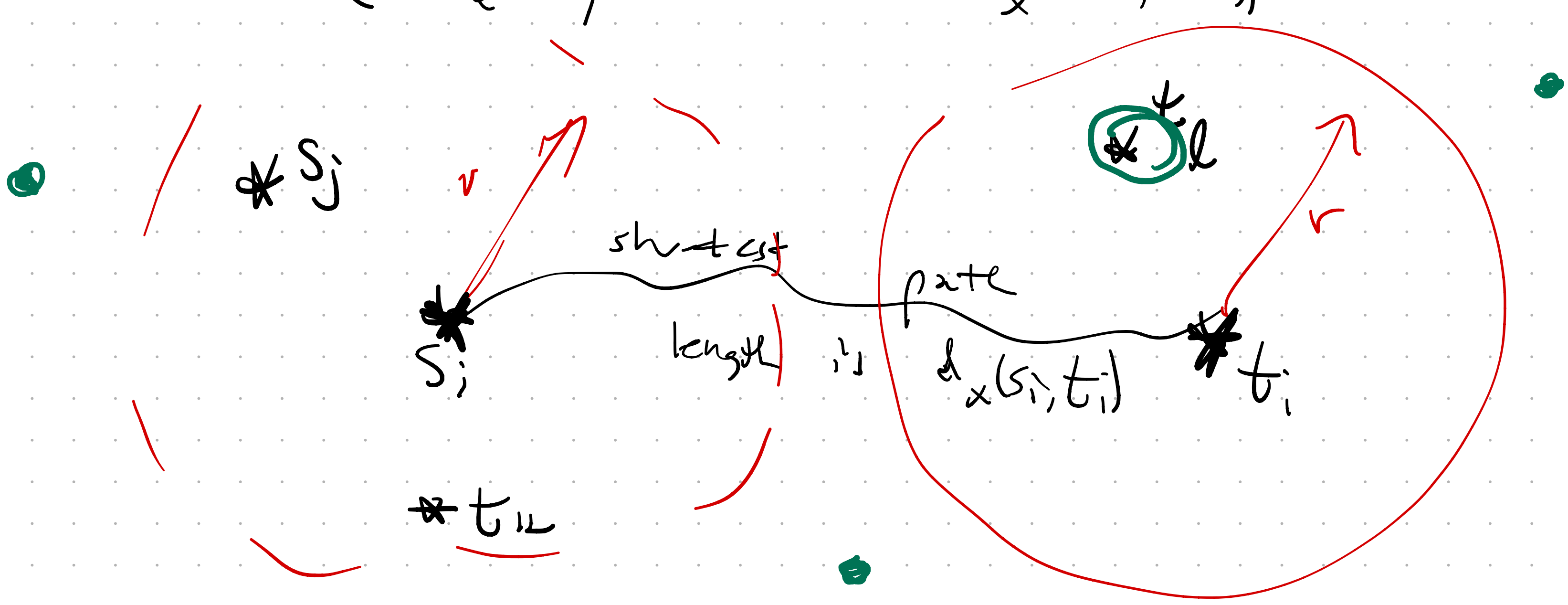
$$\geq \frac{C}{\log k} \int_0^{\frac{1}{2}\mathrm{dist}_x(s_i,t_i)} dr = \frac{C \cdot \mathrm{dist}_x(s_i,t_i)}{2 \log k}.$$

$$\mathbb{E}[\mathrm{sep}(C)] = \sum_{i=1}^{k} \Pr(C \text{ separates } s_i, t_i)$$

$$\geq \frac{C}{2 \log k} \sum_{i=1}^{k} \mathrm{dist}_x(s_i, t_i) = \frac{C}{2 \log k}.$$

Remains for us to show: For $0 < r < \frac{1}{2}\mathrm{dist}(s_i, t_i)$

$$\Pr(\mathrm{dist}_x(s_i, W) < r < \mathrm{dist}_x(t_i, W)) + \Pr(\mathrm{dist}_x(t_i, W) < r < \mathrm{dist}_x(s_i, W)) \geq \frac{C}{\log k}.$$



Let $\mathcal{U} = \left\{ u \;\middle|\; \begin{array}{l} u = s_j \text{ or } u = t_j \text{ for some } j, \\ d(u, \{s_i, t_i\}) < r \end{array} \right\}$

$$2 \leq |\mathcal{U}| \leq 2k.$$

Observe: If $|\mathcal{U} \cap W| = 1$ then either

$\text{dist}_x(s_i, W) < r < \text{dist}_x(t_i, W)$ or the reverse.

Let 2^j be the smallest power of 2 greater than or equal to $|\mathcal{U}|$.

Recall W was sampled by selecting
$p \in \{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^{\lceil \log_2(2k) \rceil}}\}$ uniformly

then each vtx in $S \cup T$ was included in W indep't prob. p.

$$\Pr\left(p = \frac{1}{2^j}\right) = \frac{1}{1 + \lceil \log_2(2k) \rceil} \geq \frac{1}{3 + \log_2(2k)} \geq \frac{1}{5 \log_2(k)}$$

$$\Pr\left(|\mathcal{U} \cap W| = 1 \;\middle|\; p = \frac{1}{2^j}\right) = \sum_{u \in \mathcal{U}} \Pr(\{u\} = \mathcal{U} \cap W)$$

$$= \underbrace{|\mathcal{U}| \cdot 2^{-j}}_{\text{by choice of } j} \cdot \underbrace{(1 - 2^{-j})^{|\mathcal{U}| - 1}}_{(1 - \frac{1}{n})^{n-1} > \frac{1}{e} \;\forall n \in \mathbb{N}}$$

$$> \frac{1}{2} \cdot \frac{1}{e}$$

$$\Pr(|U \cap W| = 1) \geq \frac{1}{10e \log k}.$$

How to sample random C s.t.

$$\mathbb{E}\left[\frac{cap(C)}{sep(C)}\right] \leq O(\log k) \cdot vol(x)$$

We already designed a "region growing" procedure that satisfies

$$\frac{\mathbb{E}[cap\ C]}{\mathbb{E}[sep\ C]} \leq O(\log k) \cdot vol(x)$$

$$\frac{\sum_C RG(C) \cdot cap(C)}{\sum_C RG(C) \cdot sep(C)}$$

$RG(C) = \Pr($region growing samples $C)$

"Pr" refers to RG

$$= \frac{\sum_C RG(C) \cdot sep(C) \cdot \left(\frac{cap(C)}{sep(C)}\right)}{\sum_C RG(C) \cdot sep(C)}$$

If $RS(C) = \dfrac{RG(C) \cdot sep(C)}{\sum_{C'} RG(C') \cdot sep(C')}$

then $\mathbb{E}\left[\frac{cap(C)}{sep(C)}\right]$ under this distribution RS is $O(\log k)$.

Rejection sampling algorithm

repeat {

sample C using RG

calculate $sep(C)$

reject C with probability $1 - \frac{sep(C)}{k}$

} until C not rejected

output C.

Why does it work?

Conditional on procedure stopping in iteration t, (for any t)

$$Pr(\text{output } C \mid \text{stop at } t)$$

$$= \frac{Pr(\text{stop at } t \text{ and output } C)}{Pr(\text{stop at } t)}$$

$$= \frac{Pr(\text{stop after } t-1) \cdot Pr\left(\begin{array}{c}C \text{ selected by RG} \\ \text{in iteration } t\end{array} \text{ and not rejected}\right)}{Pr(\text{stop at } t)}$$

$$= \frac{Pr(\text{stop after } t-1)}{Pr(\text{stop at } t)} \cdot RG(C) \cdot \left(\frac{sep(C)}{k}\right)$$

$$\propto RG(C) \cdot sep(C)$$