$\boxed{\text{30 Oct 2024}}$ Finishing approximate max-flow min-cut

Recall.

$$\text{Sparsity}(C) = \frac{\text{cap}(C) = \sum_{e \in C} c(e)}{\text{sep}(C) = \#\{i \mid s_i, t_i \text{ separated by } C\}}$$

We have seen:

$$\text{max concurrent MCF rate} = \underbrace{\text{min "fractional cut"}} \leq \min\{\text{sparsity}(C)\}$$

of (s_i, t_i) pairs

$$\leq O(\log k) \cdot \text{max concurrent MCF rate}$$

$$\min \sum_{e \in E} c(e) \cdot x_e$$

$$\text{s.t.} \quad \sum_{e \in E} n_Q(e)\, x_e \geq 1 \quad \forall Q \in \prod_{i=1}^{k} \mathcal{P}(s_i, t_i)$$

$$x_e \geq 0 \quad \forall e$$

$$\sum_{e \in E} n_Q(e)\, x_e = \sum_{e \in E} \sum_{i=1}^{k} \mathbb{1}[e \in P_i]\, x_e$$

$$= \sum_{i=1}^{k} \left(\sum_{e \in P_i} x_e \right)$$

$$= \sum_{i=1}^{k} \text{length}_x(P_i)$$

Corollary. A separation oracle can be implemented by solving k shortest-path problems to find $P_i \in \mathcal{P}(s_i, t_i)$ minimizing $\text{length}_x(P_i)$ $\forall i \in [k]$.

Procedure for randomly sampling a cut.

1. Let $S = \{s_1, \dots, s_k\}$
$T = \{t_1, \dots, t_k\}$

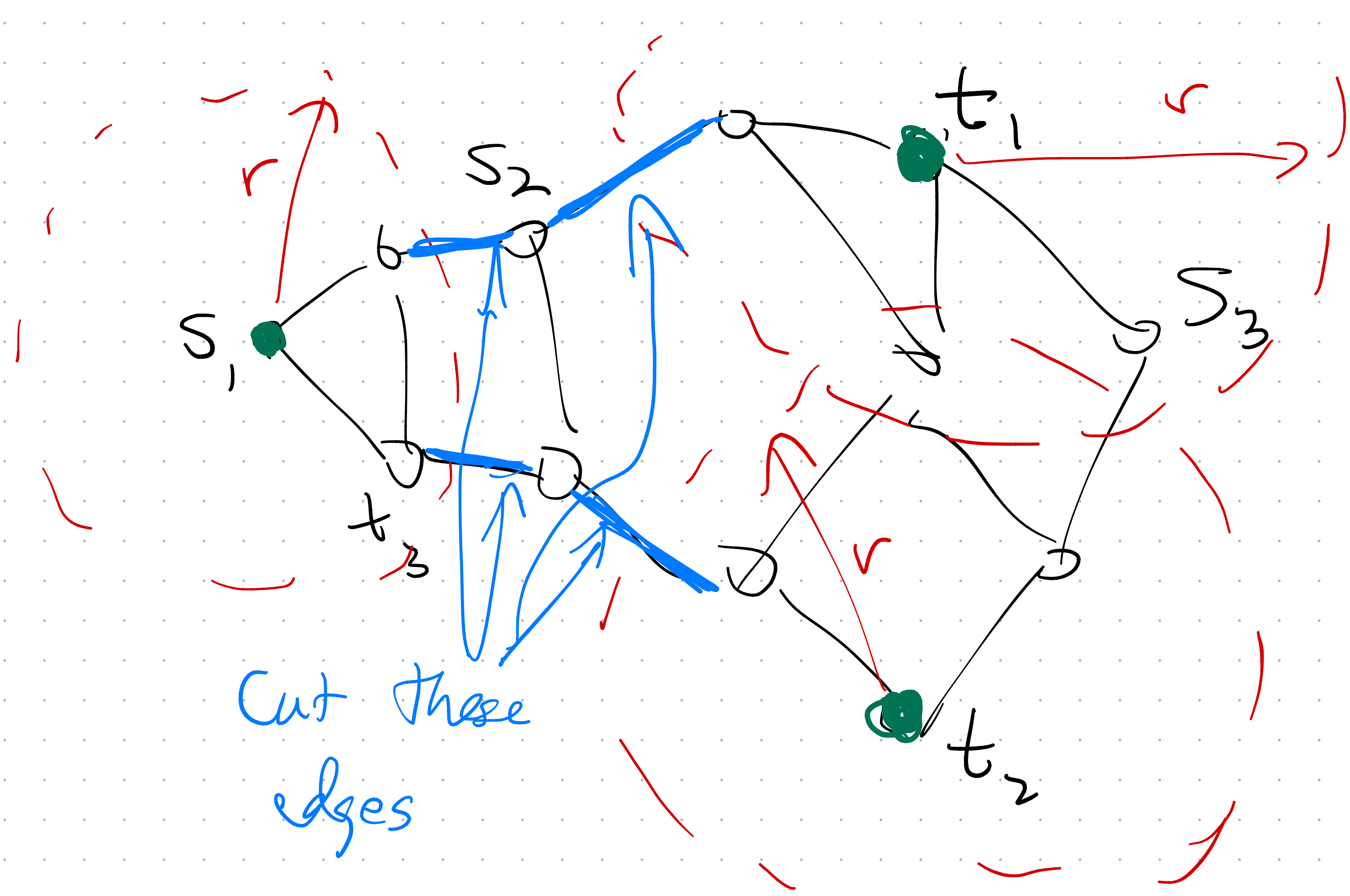
2. Choose p unif. random from
$$\left\{1, \tfrac{1}{2}, \tfrac{1}{4}, \dots, 2^{-\lceil \log_2 (2k) \rceil}\right\}$$

3. Sample $W \subseteq S \cup T$ be selecting each element of $S \cup T$ indep. with probability p.

4. Sample $r \sim \mathrm{Unif}([0,1])$.

5. Let $A = \{v \mid \mathrm{dist}_x(W, v) < r\}$
$B = V \setminus A$

output this random cut. $\rightarrow$ $C = \{\text{edges from } A \text{ to } B\}$.



$\mathrm{dist}_x(W, v)$ means

$$\min\left\{ \mathrm{length}_x(P) \;\middle|\; \begin{array}{l} P \text{ a path from } v \text{ to} \\ \text{an element of } W \end{array} \right\}$$

Analysis:

Step 1. $\mathbb{E}[\mathrm{cap}(C)] \leq \sum c(e) x_e$.

$$\mathbb{E}[\mathrm{cap}(C)] = \sum_e c(e) \Pr(e \in C)$$

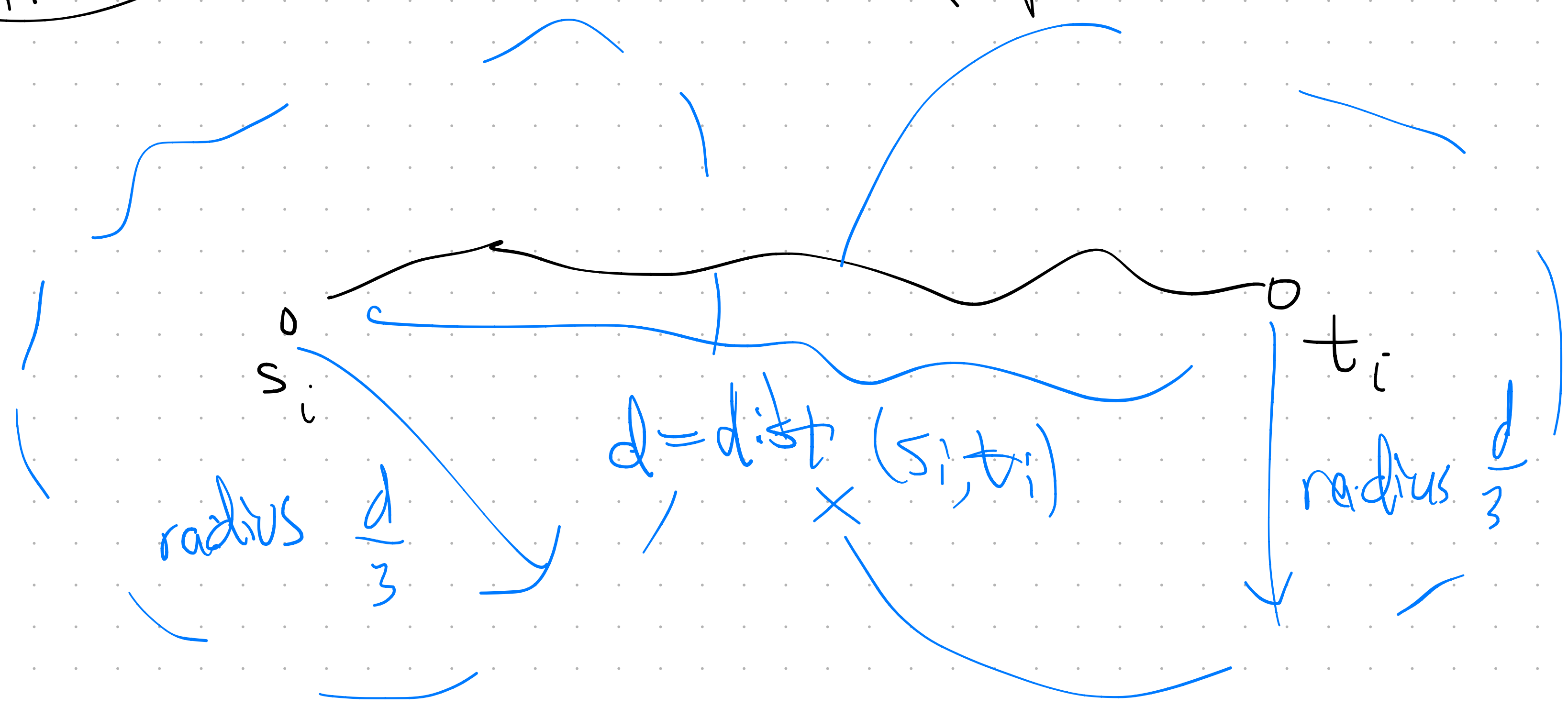
If $e = (u, v)$ then the event $e \in C$ happens if and only if radius r lies in the interval from $\mathrm{dist}_x(W, u)$ to $\mathrm{dist}_x(W, v)$.

$$\Pr(e \in C) = \left| \mathrm{dist}_x(W, u) - \mathrm{dist}_x(W, v) \right|$$

$$\leq x_e.$$

Step 2: $E[\mathrm{sep}(C)] \geq \frac{1}{O(\log k)}$

Proof: Focus on $\Pr(\text{separate } s_i \text{ from } t_i)$.



Def $\mathcal{U} = \left\{ u \in S \cup T \;\middle|\; \mathrm{dist}_x(u, \{s_i, t_i\}) < \frac{d}{3} \right\}$