Chernoff bound

$X_1 \ldots X_n$ indep $\quad X_i \in [0,1]$

$X = \sum_i X_i \qquad \mu = E(X)$

$\Pr(X > \beta\mu) \le e^{-\mu(\beta \ln\beta - \beta + 1)}$ for $\beta > 1$

$\Pr(X < \beta\mu) \le e^{-\mu(\beta \ln\beta - \beta + 1)}$ for $\beta < 1$

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how good is this

$\beta = 1 + \delta \qquad \ln(1+\delta) \sim \delta$

$\qquad\qquad (\text{e.g. } e^{\delta} = 1 + \delta + \frac{\delta^2}{\ldots}$

$\beta > 1$ case bound

$$e^{-\mu\left[(1+\delta)\delta - (1+\delta) + 1\right)}$$

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$\beta < 1$ case

$\beta = 1 - \delta \qquad \ln(1-\delta) \sim -\delta$

bound $\qquad e^{-\mu\left[(1-\delta)\cdot(-\delta) - (1-\delta) + 1\right]}$

Proving the bound

$$\delta \cdot \Pr(Y \geq \delta) \leq E(Y)$$

$Y \geq 0$ variable, Markov

~~If~~: Use this e^{tX}

if $t > 0$: $X > \beta\mu$ iff $e^{tX} > e^{t\mu\beta}$

if $t < 0$: $X < \beta\mu$ iff $e^{tX} > e^{t\mu\beta}$

What is

$$E(e^{tX}) = E\left(e^{t\sum_i X_i}\right)$$

$$= E\left(\prod_i e^{tX_i}\right) = \prod_i E\left(e^{tX_i}\right)$$

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assume $E(X_i) = \mu_i$ $\left(\mu = \sum \mu_i\right)$

Claim Y_i $\Pr(Y_i=1)=\mu_i$ & 0 otw

$$E(e^{tY_i}) \geq E(e^{tX_i})$$

flip X_i then 0/1 w/ prob X_i to 1

[if X_i takes finite options]

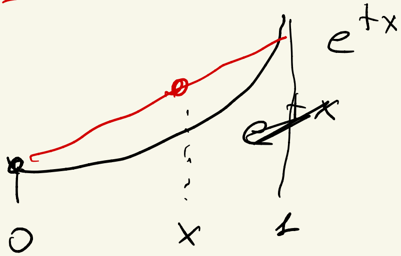
$\Rightarrow$ prob new coin 1

$$\sum_x \Pr(X_i=x)\cdot X_i = \mu_i$$

$$E(e^{tY_i} \mid X_i=x) \geq E(e^{tX_i} \mid X_i=x)$$

$$e^t x + 1(1-x) \geq e^{tx}$$

$\Leftrightarrow$



adding all $X_i=x$

$$E(e^{tY_i}) \geq E(e^{tX_i})$$

Using this

$$E(e^{tY_i}) = e^t \mu_i + 1(1-\mu_i)$$
$$= 1 + \mu_i(e^t - 1) \leq e^{\mu_i(e^t-1)}$$

Putting it together

$$E(e^{tX}) = \prod_i E(e^{tX_i}) \leq \prod_i E(e^{tY_i})$$
$$\leq \prod_i e^{\mu_i(e^t-1)} = e^{\mu(e^t-1)}$$

Using Markov

$\beta > 1$ $\quad Pr(X > \beta\mu) = Pr(e^{tX} > e^{\beta\mu t})$

$t > 0$

$$\leq \frac{E(e^{tX})}{e^{\beta\mu t}} \leq e^{\mu(e^t-1)-\beta\mu t}$$

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if $\beta < 1$ $\quad Pr(X < \beta\mu) = Pr(e^{tX} > e^{\beta\mu t})$

$t < 0$

$$\leq \frac{E(e^{tX})}{e^{\beta\mu t}} \leq e^{\mu(e^t-1)-\beta\mu t}$$

Made with Goodnotes

Optimizing μ of exponent

$$\mu \cdot e^t - \beta\mu \implies \text{set } t = \ln\beta$$

substituting into result:

bound is

$$e^{\mu(\beta-1) - \beta\mu \cdot \ln\beta}$$

Example $X_1 \dots X_n$ 0/1

$$E(X_i) = \frac{1}{n} \implies E(\sum X_i) = 1$$

Question

$$\Pr(\sum X_i > \alpha) \leq \frac{1}{N}$$

Our bond gives Pr estimate

$$e^{(\alpha-1) - \alpha\ln\alpha} < \frac{1}{N}$$

* <u>Claim:</u> $\alpha > \frac{\ln N}{\ln\ln N}$