$\boxed{\text{23 Oct 2024}}$ Approximate max-flow min-cut for Sparsest cut

MCMF LP

$$\max \sum_{Q \in \prod_{i=1}^{k} P_i} y_Q$$

$$\text{s.t. } \sum_{Q} n_Q(e)\, y_Q \le c(e) \quad \forall e$$

$$y \geq 0$$

DUAL

$$\min \sum_e c(e)\, x_e$$

$$\text{s.t. } \sum_e n_Q(e)\, x_e \ge 1 \quad \forall Q$$

$$x \geq 0$$

An x that is feasible for DUAL is a "fractional cut."

If $C \subset E$ is an edge set that separates p terminal pairs (s_i, t_i) consider this vector:

$$x_e = \begin{cases} \frac{1}{p} & \text{if } e \in C \\ 0 & \text{if } e \notin C. \end{cases}$$

This is feasible for the dual, and the dual objective

$$\sum_e c(e)\, x_e = \frac{1}{p} \sum_{e \in C} c(e)$$

$$= \frac{1}{p} \cdot \text{capacity}(C).$$

$$= \text{sparsity}(C)$$

Strong duality:

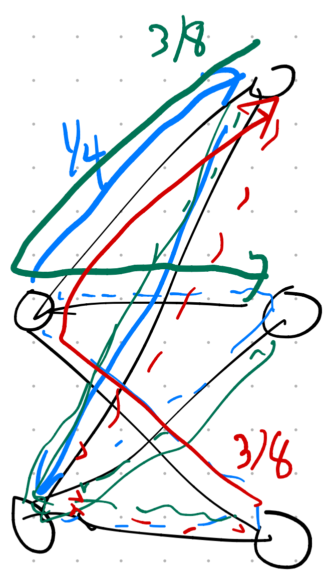
$$\mathrm{OPT}(\text{MCMF LP}) = \mathrm{OPT}(\text{DUAL LP})$$

$\mathrm{OPT}(\text{MCMF LP})$ $=$ max concurrent flow rate $\Uparrow$ can efficiently compute this

$\mathrm{OPT}(\text{DUAL LP})$ $\leq$ sparsest cut value $\Updownarrow$ NP-hard to compute

The Okamura-Seymour example

$\frac{1}{4}+\frac{3}{8}+\frac{3}{8}=1$



Edges $c(e)=1$ for all e.

$k=4$

Every (s_i,t_i) s.t. s_i,t_i are on the same side of the graph as one another

$\boxed{r=3/4}$ is the max concurrent flow rate.

The optimal dual is $x_e=\frac{1}{8}$ for all e.

$$\sum c(e)x_e = 6\cdot\left(\frac{1}{8}\right)=\frac{3}{4}.$$

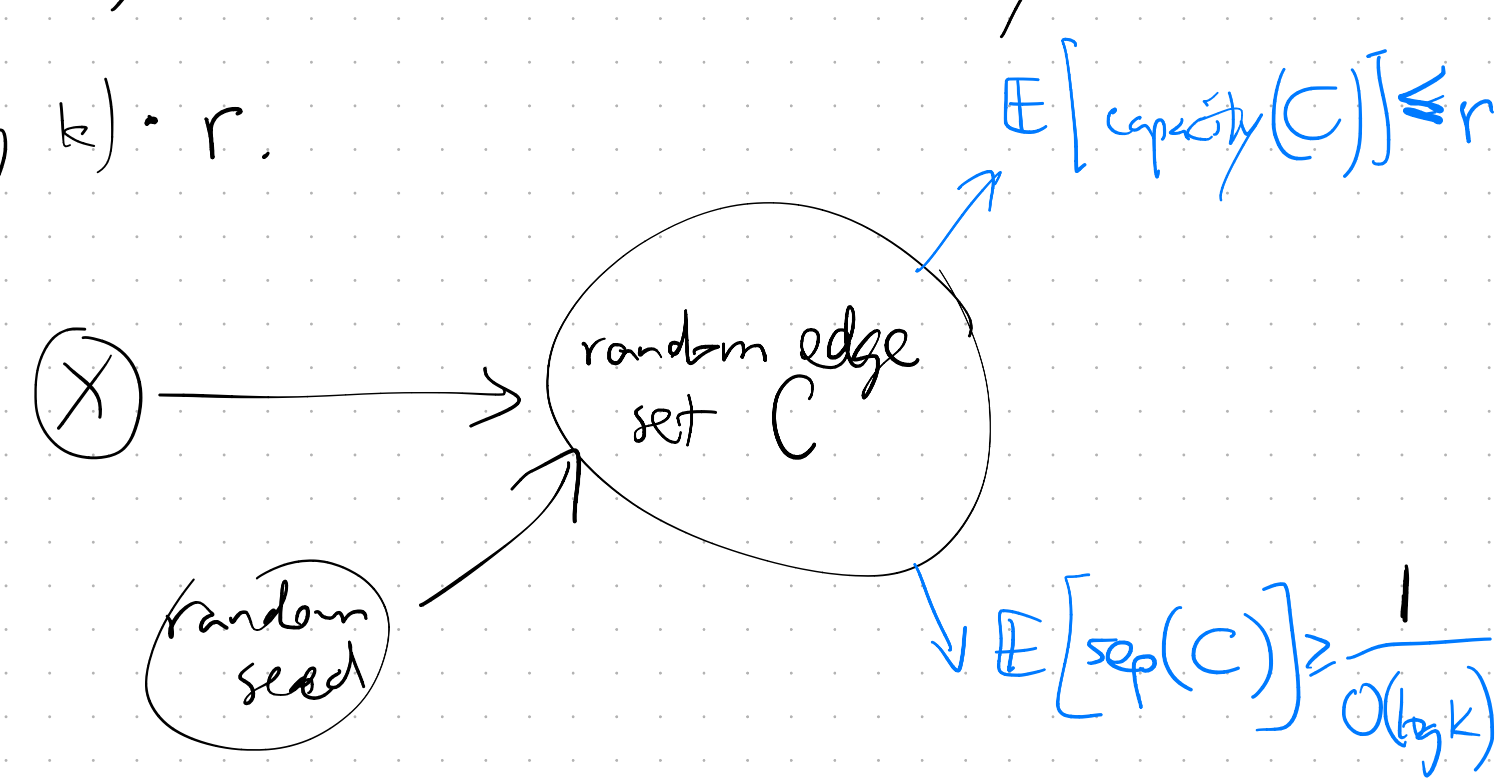
Sparsest cut value $=1$ for this graph.

Theorem. For any undirected instance of MAX CONCURRENT MULTICOMMODITY FLOW, the ratio $\frac{\text{sparsest cut}}{\text{max conc. flow rate}}$ is $O(\log k)$. (k = # of terminal pairs)

Proof. Solve DUAL to obtain fractional cut x such that

$$\sum c(e) x_e = r = \text{max concurrent flow rate.}$$

Plan is to randomly round x to an edge cut whose sparsity is $O(\log k) \cdot r$.



This will imply $\exists\, C^*$ such that

$$\frac{\text{capacity}(C^*)}{\text{sep}(C^*)} \leq r \cdot O(\log k).$$

Proof by contradiction ....

Suppose $\forall C$ $\quad capacity(C) > r \cdot O(\log k) \cdot sep(C)$.

Averaging, $\quad \mathbb{E}[capacity(C)] > r \cdot O(\log k) \cdot \mathbb{E}[sep(C)]$

r $\qquad\qquad \geq \frac{1}{O(\log k)}$

Sampling strategy: let W be a <u>random</u> subset of $\{s_1, \dots, s_k\} \cup \{t_1, \dots, t_k\}$.

The distribution of W will be tricky!

Sample $b \in \{1, 2, 4, 8, \dots, 2^{\lceil \log_2(2k) \rceil}\}$ uniformly.

Each $w \in \{s_1, \dots, s_k\} \cup \{t_1, \dots, t_k\}$ independently with probability $1/b$ decides to join W.

For each $v \in V(G)$ let

$$d(v, W) = \min \left\{ \sum_{e \in P} x_e \;\middle|\; \begin{array}{l} P \text{ a path from} \\ v \text{ to } w \in W \end{array} \right\},$$

Sample $t \in [0,1]$ unif random.

Cut edge $e = (u, v)$ if and only if

$$d(u, W) < t < d(v, W)$$

or $$d(v, W) < t < d(u, W).$$