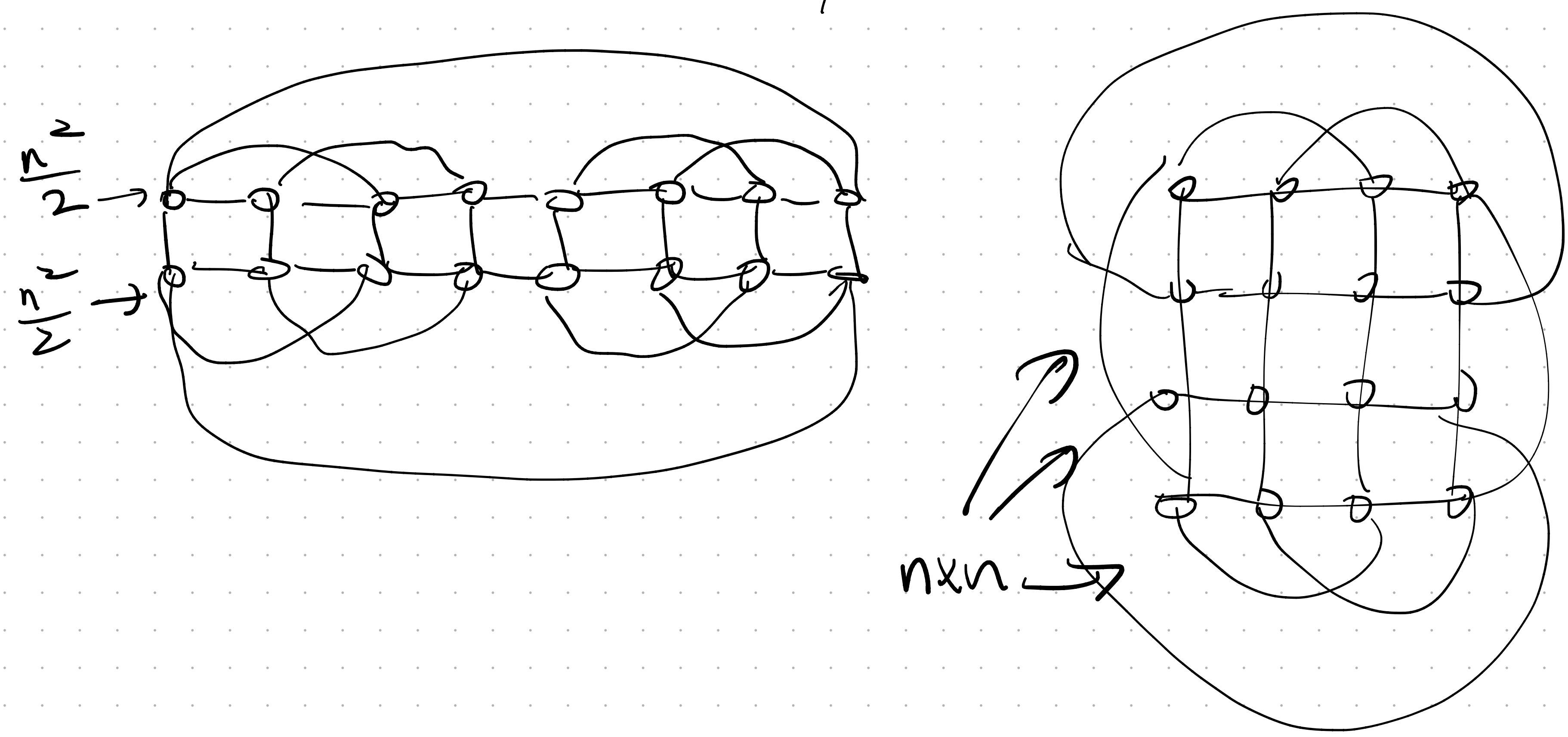
21 Oct 2024

Introducing sparsest cut and multicommodity flow



Which of the graphs is "easier" to cut into two pieces?

Def. The <u>sparsity</u> of a cut (A, B) in a graph G is:

$$\frac{\mathrm{cap}(A,B)}{\min\{\mathrm{vol}(A), \mathrm{vol}(B)\}}$$

where $\mathrm{vol}(S) = \sum_{v \in S} \mathrm{degree}(v)$

The (uniform) sparsest cut problem asks for a vertex set $S \neq \emptyset, V$ that minimizes $\mathrm{sparsity}(S, V-S)$.

Generalization of sparsest cut:

We have k source-dest pairs. $\{(s_i, t_i)\}_{i=1}^{k}$

We're looking for an edge set C that separates at least one pair.

$$\text{Sparsity}(C) = \frac{\sum_{e \in C} \text{capacity}(e)}{\#\{i \mid \text{every } s_i\text{-}t_i \text{ path intersects } C\}}.$$

E.g. if G is d-regular with n vertices and $k = \binom{n}{2}$

and every pair of vertices is an s_i-t_i pair

and $C = \{\text{edges from } A \text{ to } B\}$

then

$$\text{sparsity}(C) = \frac{c(A, B)}{|A| \cdot |B|}$$

$$= d^2 \cdot \frac{c(A,B)}{\text{vol}(A) \cdot \text{vol}(B)}$$

$$= d^2 \frac{c(A,B)}{\min\{\text{vol}(A), \text{vol}(B)\} \cdot \max\{\text{vol}(A), \text{vol}(B)\}}$$

Both $\text{vol}(A), \text{vol}(B) \leq \text{vol}(V)$ and

$\text{vol}(A) + \text{vol}(B) = \text{vol}(V)$

So $\max\{\text{vol}(A), \text{vol}(B)\} \geq \frac{1}{2} \text{vol}(V)$.

So minimizing sparsity(C) minimizes the other sparsity objective within a factor of 2.

For $k=1$ this is the s-t min cut problem.

Max Concurrent Multicommodity Flow

Simultaneously find a flow of rate (i.e. value) r from every s_i to t_i, while respecting capacity constraints on edges.

Maximize r.

paths from s_i to t_i

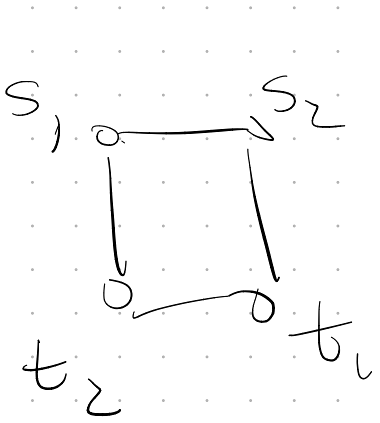
Let $$\mathcal{P} = \prod_{i=1}^{k} \mathcal{P}(s_i, t_i)$$

number of paths passing through e in the k-tuple Q.

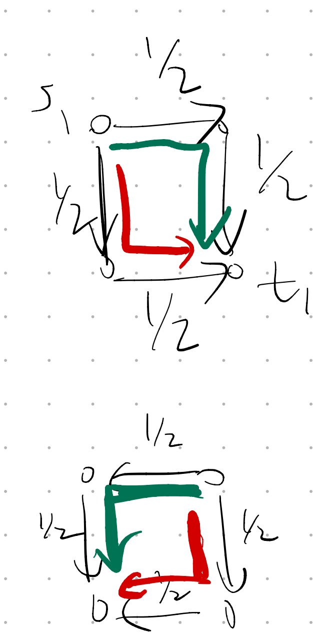
$$\max \sum_{Q \in \mathcal{P}} y_Q$$

$$\text{s.t.} \quad \sum_{Q \in \mathcal{P}} \boxed{n_Q(e)}\, y_Q \leq c(e) \quad \forall\, e \in E$$

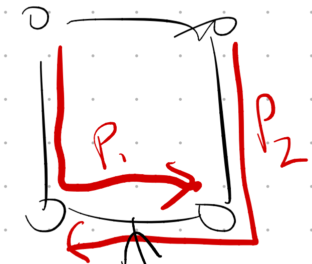
$$y_Q \geq 0 \quad \forall\, Q \in \mathcal{P}.$$



Achieving rate 1:

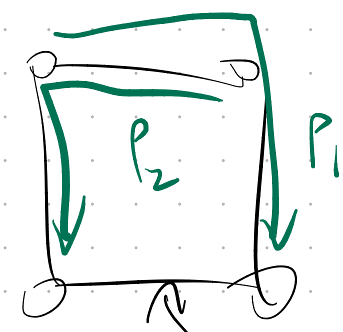


Q_1



$n_{Q_1}(e) = 2$

Q_2



$n_{Q_2}(e) = 0$

$$y_Q = \begin{cases} 1/2 & \text{if } Q = Q_1 \\ 1/2 & \text{if } Q = Q_2 \\ 0 & \text{otherwise} \end{cases}$$

$$\max \sum_{Q \in \mathcal{P}} y_Q$$

$$\text{s.t.} \quad \sum_{Q \in \mathcal{P}} n_Q(e) y_Q \leq c(e)$$

$$y_Q \geq 0$$

Dual of MCMF: $\min \sum_e c(e) x_e$

$$\text{s.t.} \quad \sum_e n_Q(e) x_e \geq 1 \quad \forall Q \in \mathcal{P}$$

$$x_e \geq 0 \quad \forall e$$

A solution $\vec{x}$ feasible for the above dual LP will be called a "fractional cut".