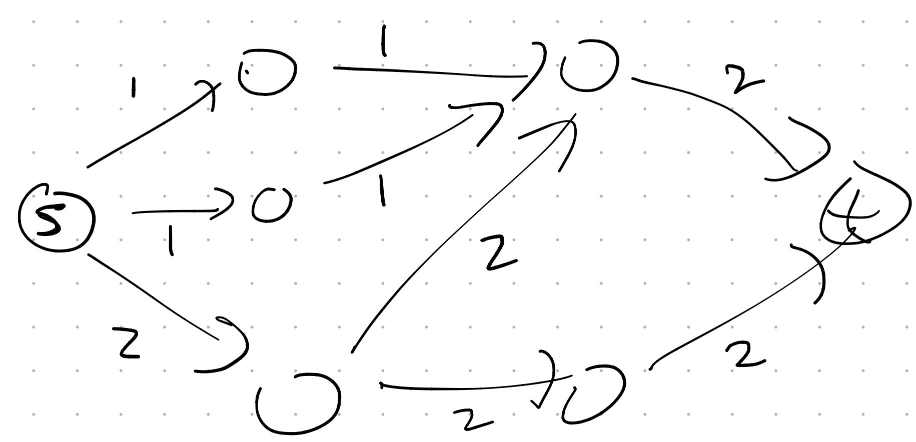
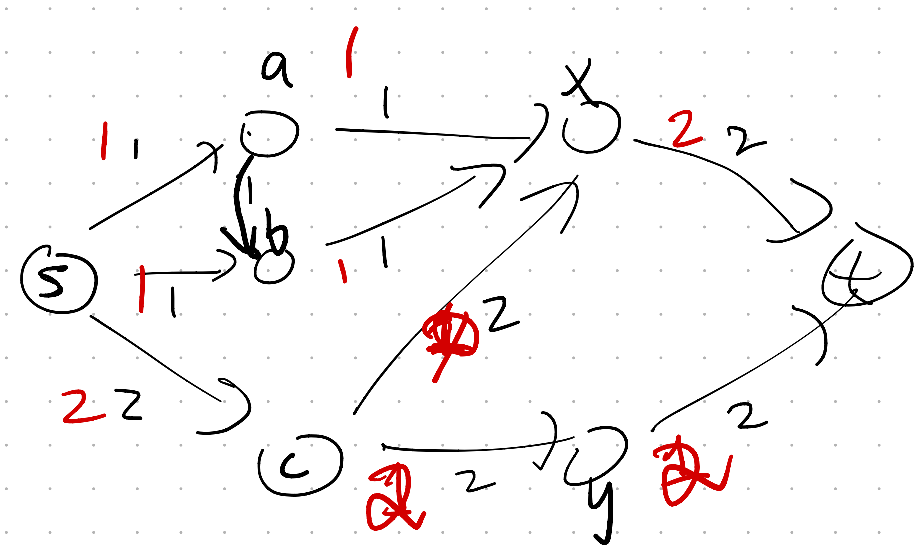
18 Oct 2024

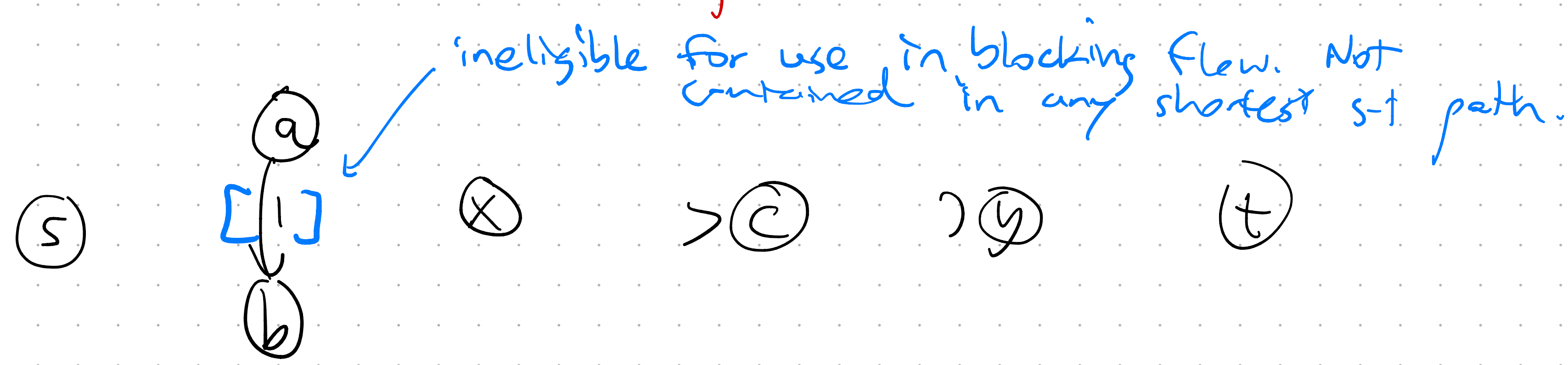
Applications of Max-Flow Min-Cut.

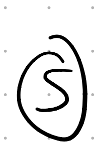
Finishing Dinitz ...





End of 1st phase





$O(mn)$ time per blocking flow because each augmenting path takes $O(n)$ time to fill up the stack, and $\leq m$ augmenting paths are found in the blocking flow. (Each saturates a distinct edge.)

(1) Max-Flow Min-Cut Theorem

(2) Flow Integrality Theorem (a network with integer capacities has an integer max flow).

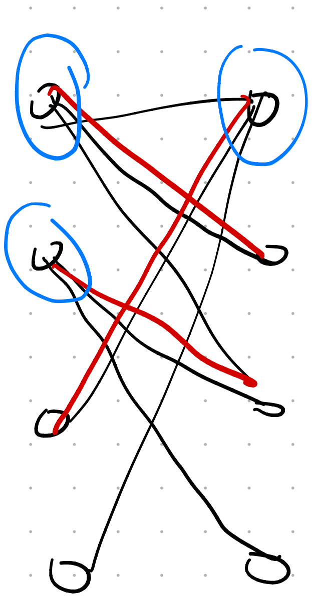
Def. A *vertex cover* of a graph G is a set of vertices such that every edge has at least one endpoint in the set.

König-Egervary Theorem If G is bipartite,

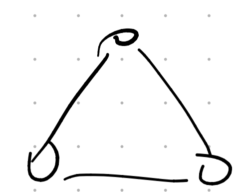
min vertex cover size of G = maximum matching size of G.

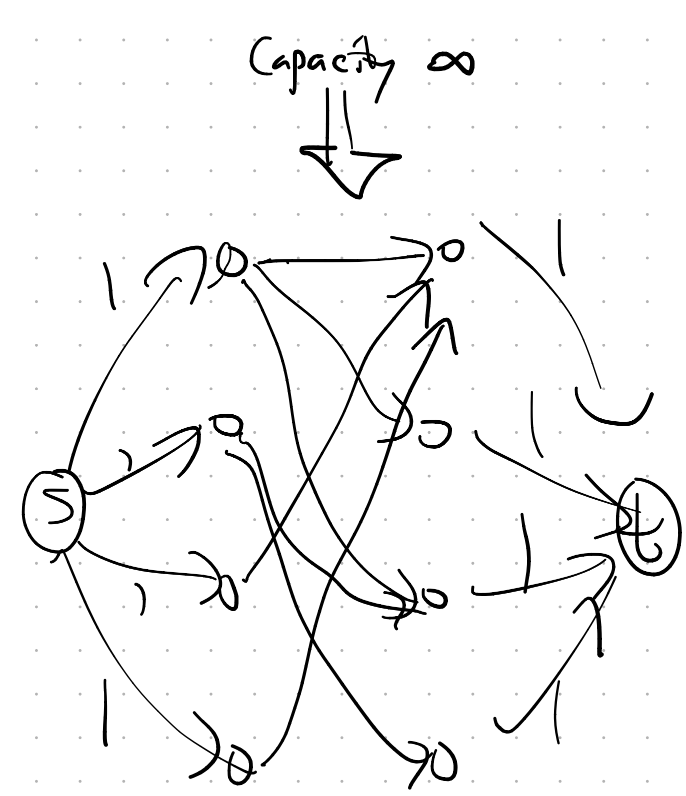
Remark For all G, whether or not bipartite,

min vertex cover size of G $\geq$ maximum matching size of G.



$2 > 1$





Proof of König-Egervary:

Max matching size $=$ max flow

$=$ min cut capacity

$= c(A, B)$

for some cut with $s \in A$, $t \in B$.

Write $A = \{s\} \cup A_L \cup A_R$ $A_L \subseteq L$, $A_R \subseteq R$

$B = \{t\} \cup B_L \cup B_R$

$$c(A,B) = |B_L| + \infty \cdot \underbrace{(\#\text{ edges from } A_L \text{ to } B_R)}_{=0} + |A_R|$$

Every edge (u,v) of the bipartite graph must satisfy

$u \notin A_L$ or $v \notin B_R$

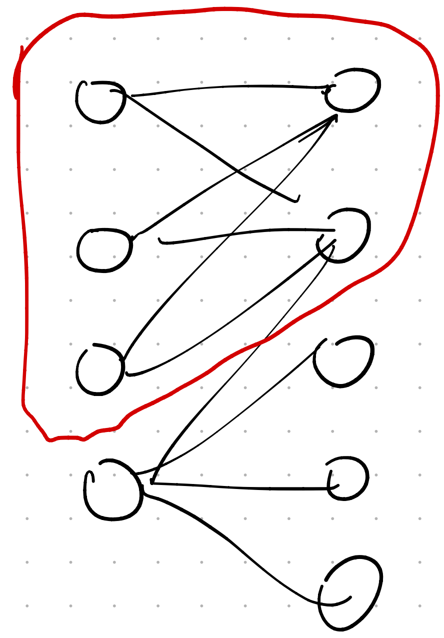
$\iff u \in B_L$ or $v \in A_R$

So $B_L \cup A_R$ is a vertex cover.

1. Hall's Marriage Theorem

$G = (V,E)$ a bipartite graph, $V = L \cup R$.

There exists a matching of size $|L|$ if and only if for every subset of L, the number of distinct neighbors is greater than or equal to the number of elements.



2. Menger's Theorems

For a (directed or undirected) graph G, and vertices s & t, the max. # of

- ⓐ edge-disjoint
- ⓑ internally vertex-disjoint

paths from s to t equals the minimum number of

- ⓐ edges
- ⓑ vertices other than s and t

that one must delete from G to disconnect s from t.