16 Oct 2024 Strongly Polynomial-Time Max Flow

Recall: Ford-Fulkerson computes a max flow if it terminates.

Observe. If all edges have non-negative integer capacities, then it always does terminate.

Proof. All flow values $f(u,v)$ and residual capacities $c_f(u,v)$ will always be integers. (Induction)

val(f) increases by at least 1 per iteration.

iterations $\leq$ val(max flow).

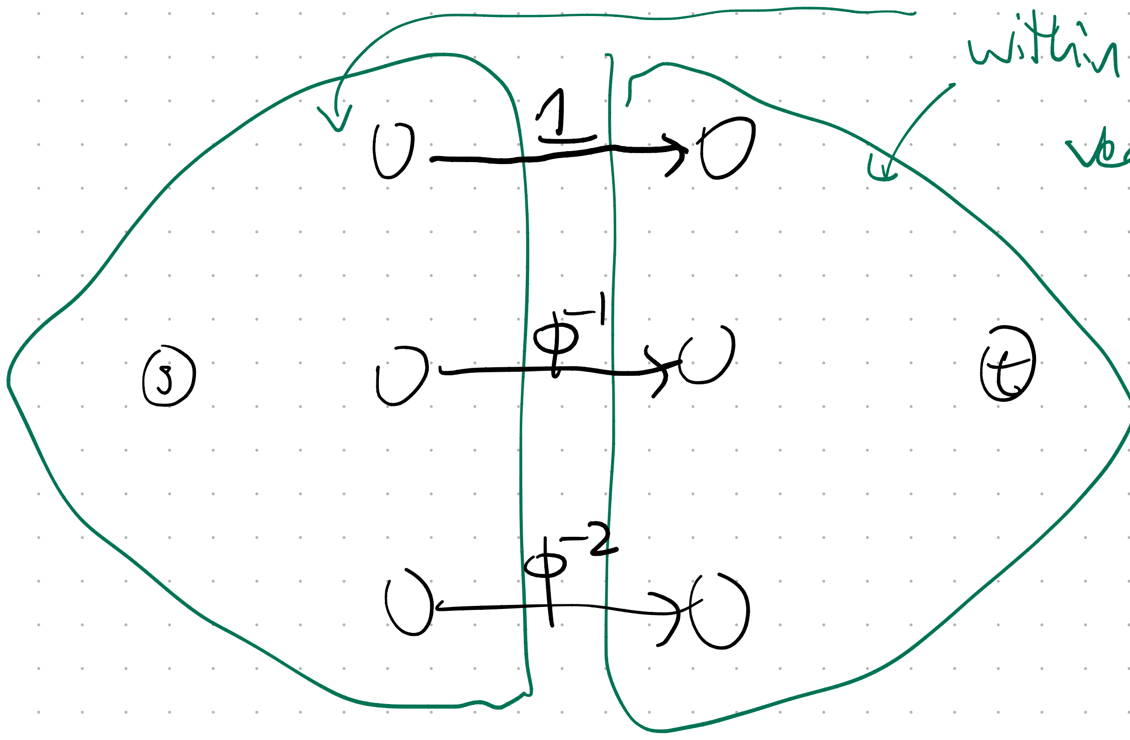
Obs. 2. If all edge capacities are rational numbers, Ford-Fulkerson always terminates.

Suppose all capacities are integer multiples of $\frac{1}{D}$ for some $D<\infty$. Repeat the argument above $\Longrightarrow$ # iters $\leq D \cdot$ val(max flow).

Obs. 3 With irrational capacities, you can have a non-terminating execution.

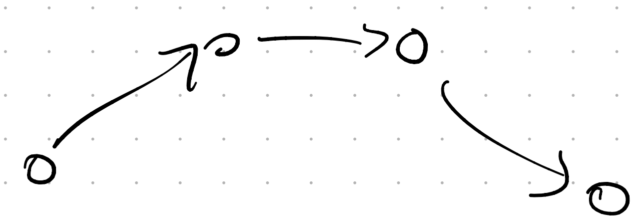
E.g. let ϕ^{-1} be the positive solution of $x^2+x=1$.

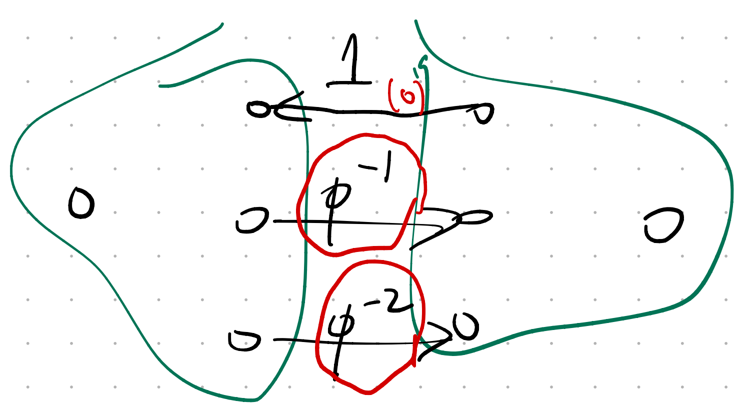
$$\phi^{-1} = \frac{\sqrt{5}-1}{2}.$$



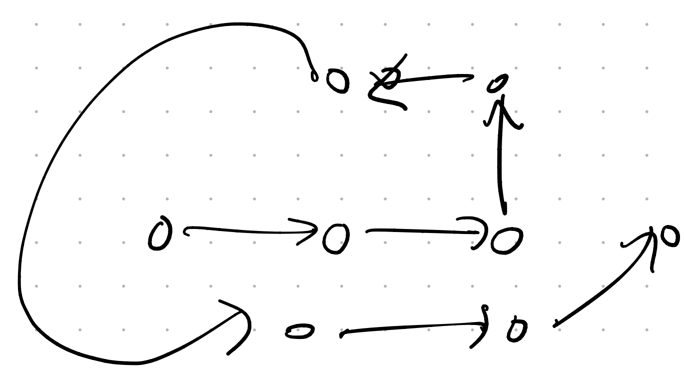
within blobs, all pairs of vertices connected with capacity ∞.

Aug path #1



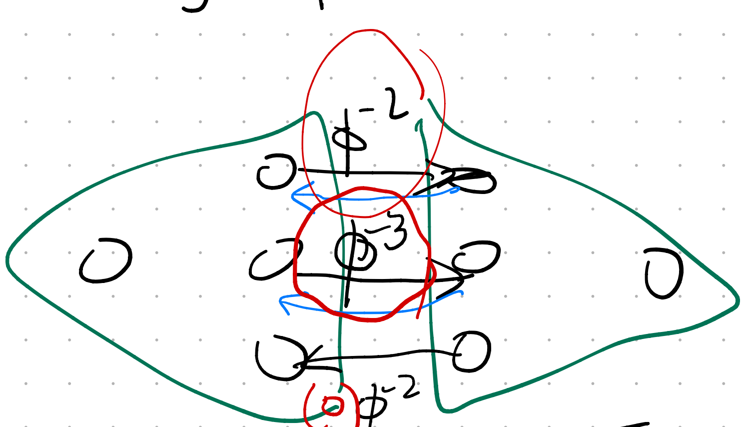


Resid graph #1



Aug path #2

$\phi^{-1} - \phi^{-2} = \phi^{-3}$



Resid graph #2

This is the same as residual graph #1 up to scaling residual capacities by ϕ^{-1} and cyclically permuting.

Idea #1. Take the first aug. path found by BFS. (Fewest edges.)

"Second Edmonds–Karp Heuristic"

$O(m^2 n)$. $m =$ # edges
$n =$ # vertices

Idea #2. "First Edmonds–Karp Heuristic"

Use the augmenting path with greatest bottleneck capacity.

$O(m^2 \log(n) \log(U))$

~~for~~ integer capacities in the range $[1, U]$.

(Yefim?)
Dinitz Algorithm (Analogous to Hopcroft–Karp.)

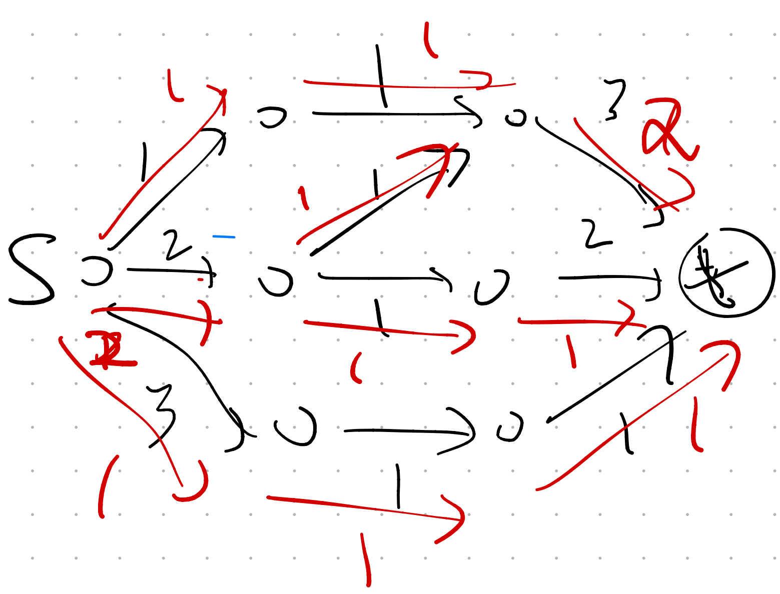
$O(mn^2)$ running time.

[$O(n)$ main loop iterations.
$O(mn)$ per iteration.]

Def. A blocking flow with respect to f is a feasible flow in G_f such that for all shortest s-t paths in G_f (those with min # of edges) at least one edge of the path is saturated by the blocking flow.

IF blocking flow denoted by b, to say b saturates (u,v) means

$b(u,v) = c_f(u,v)$, equivalently $b(u,v) + f(u,v) = c(u,v)$.



Dinitz Alg.

$f = 0$

while G_f has an s-t path:

  let b = a blocking flow w.r.t. f

  $f \leftarrow f + b$

endwhile

output f.

Lemma. The min augmenting path length in G_f strictly increases with each Dinitz iteration.