$\boxed{\text{11 Oct 2024}}$ Ford – Fulkerson

Recall. G directed graph

s, t = source, sink

$\overleftrightarrow{E} = \{ (u,v) \mid (u,v) \in E \text{ or } (v,u) \in E \}$

$f : \overleftrightarrow{E} \to \mathbb{R}$ is an s-t flow if

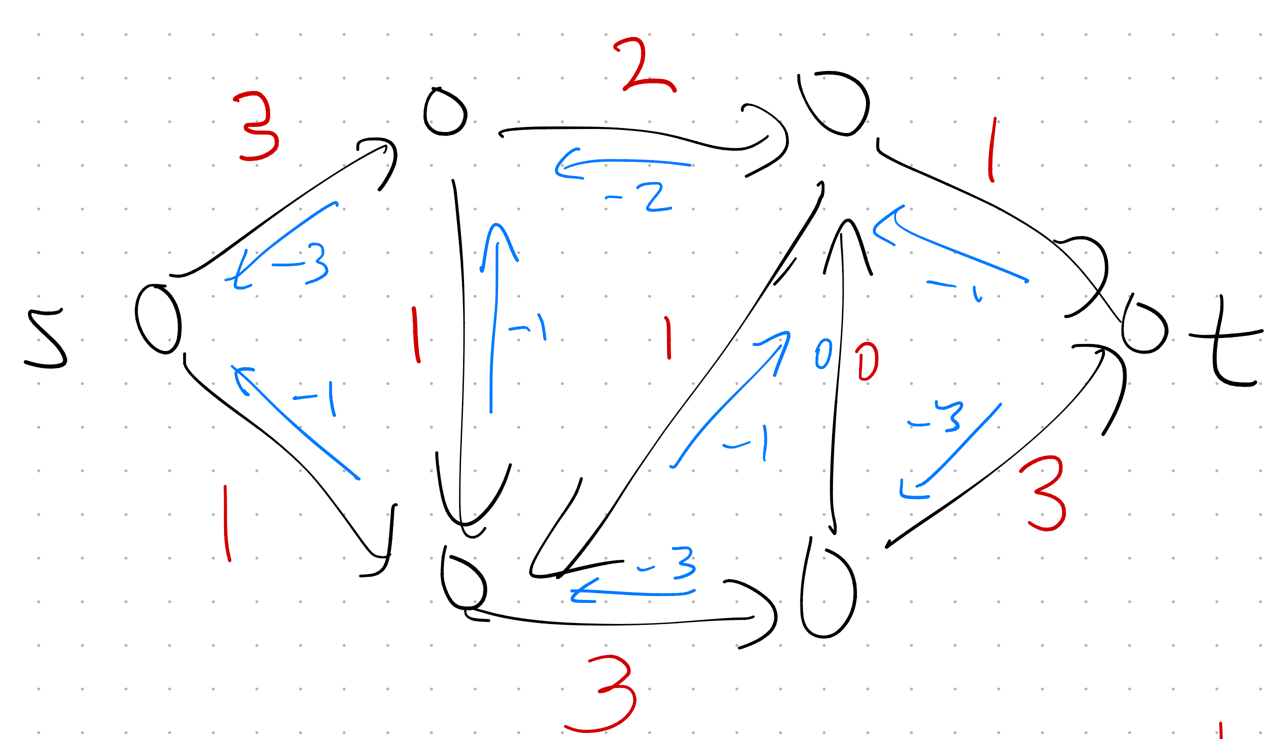
(i) $f(v,u) = -f(u,v) \quad \forall (u,v) \in \overleftrightarrow{E}$

(ii) $\sum_v f(u,v) = 0 \quad \forall u \neq s,t$

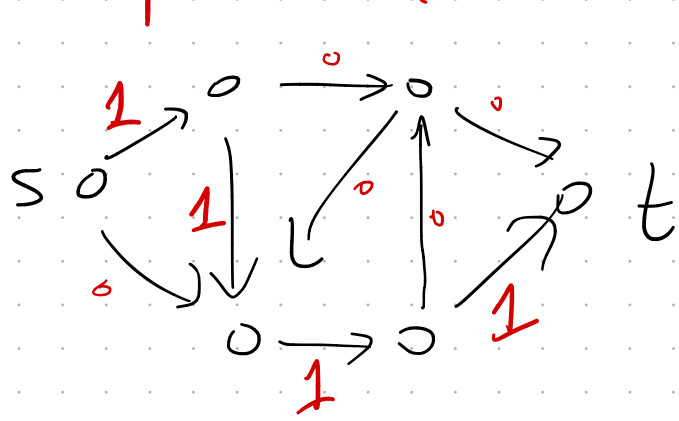
$\mathrm{val}(f) = \sum_v f(s,v)$

"feasible" means $f(e) \le c(e) \; \forall e$.

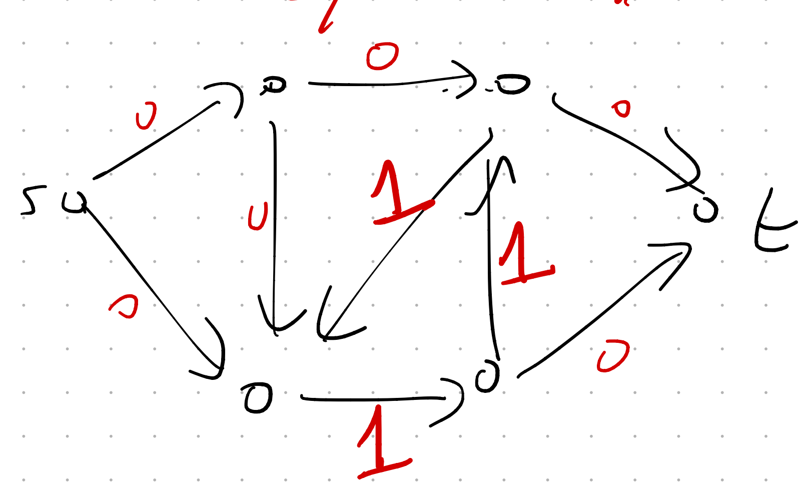
Max-Flow problem: find feasible f of maximum value.



"path flow"



"cycle flow"



Lemma: Every flow f in a graph is a weighted sum of s-t path flows and cycle flows.

$\mathrm{val}(f)$ is the sum of the s-t path weights.

Residual graph of a flow

If G is a dir graph with s, t and $c : \overleftrightarrow{E} \to \mathbb{R}_{\geq 0}$ is a capacity function "flow network"

and f is a feasible flow in G,

the residual graph G_f has capacity function $c_f : \overleftrightarrow{E} \to \mathbb{R}$

$$c_f(e) = c(e) - f(e)$$

and $V(G_f) = V(G)$, $E(G_f) = \{ e \in \overleftrightarrow{E} \mid c_f(e) > 0 \}$.

Lemma. If G is a flow network, f a feasible flow, and G_f its residual graph, there is a bijection

$$\{s\text{-}t \text{ flows in } G_f\} \longleftrightarrow \{s\text{-}t \text{ flows in } G\}$$

$$\tilde{f} \longmapsto f + \tilde{f}$$

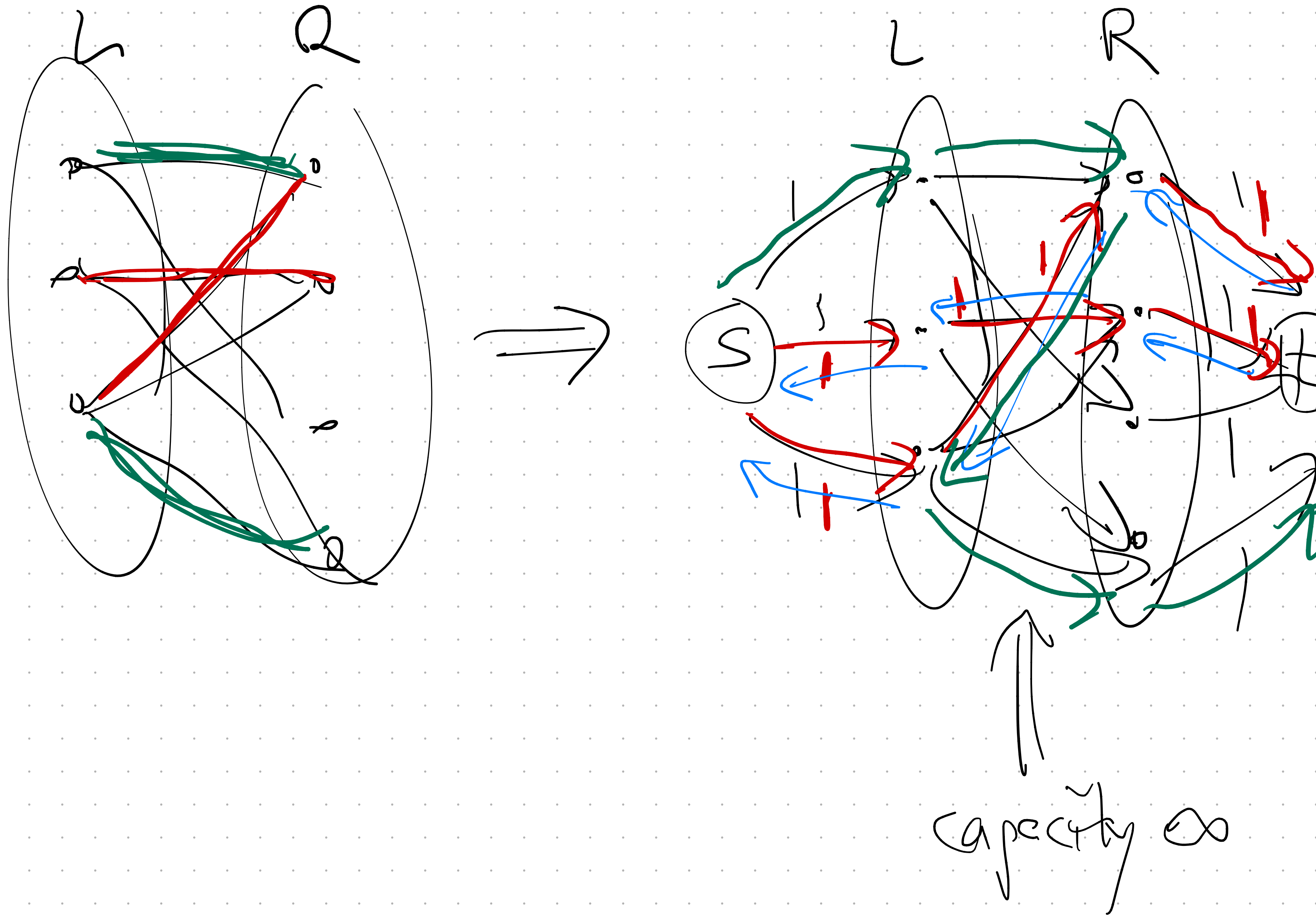
$$\mathrm{val}(\tilde{f}) + \mathrm{val}(f) = \mathrm{val}(f + \tilde{f}).$$

$$\{\text{Max-flows in } G_f\} \longleftrightarrow \{\text{Max-flows in } G\}$$

under this correspondence.

Def. An f-augmenting path is an s-t path in G_f.

Remark: Bipartite max matching reduces to max flow via...



Ford-Fulkerson Algorithm

Initialize $f(u,v) = 0 \quad \forall u,v.$

while G_f contains an s-t path P

  let $b(P) = \min\{c_f(e) \mid e \in E(P)\}$

  let f_p be a path flow on P, of value 1.

  $f \leftarrow f + b(P) \cdot f_p$

Output f.

Proposition. If Ford–Fulkerson terminates, it outputs a maximum flow.

Proof. At termination G_f has no path from s to t,

$$\{\text{max flows in } G_f\} \longleftrightarrow \{\text{max flows in } G\}$$

$$\tilde{f} = 0 \longleftrightarrow f$$

Let $A = \{\text{vertices reachable from } s \text{ in } G_f\}$

$$B = V(G_f) \setminus A,$$

This is an s-t cut.

There are no edges crossing it, so its capacity is $\emptyset$.

$\therefore$ max flow value in G_f

equals $\emptyset$

$\Longrightarrow$ $\tilde{f} = 0$ is a

max flow in G_f.