9 Oct 2024 Max Flow

RECALL. Exact s-t cut set $= \{(u,v) \mid u \in A,\ v \in B\}$

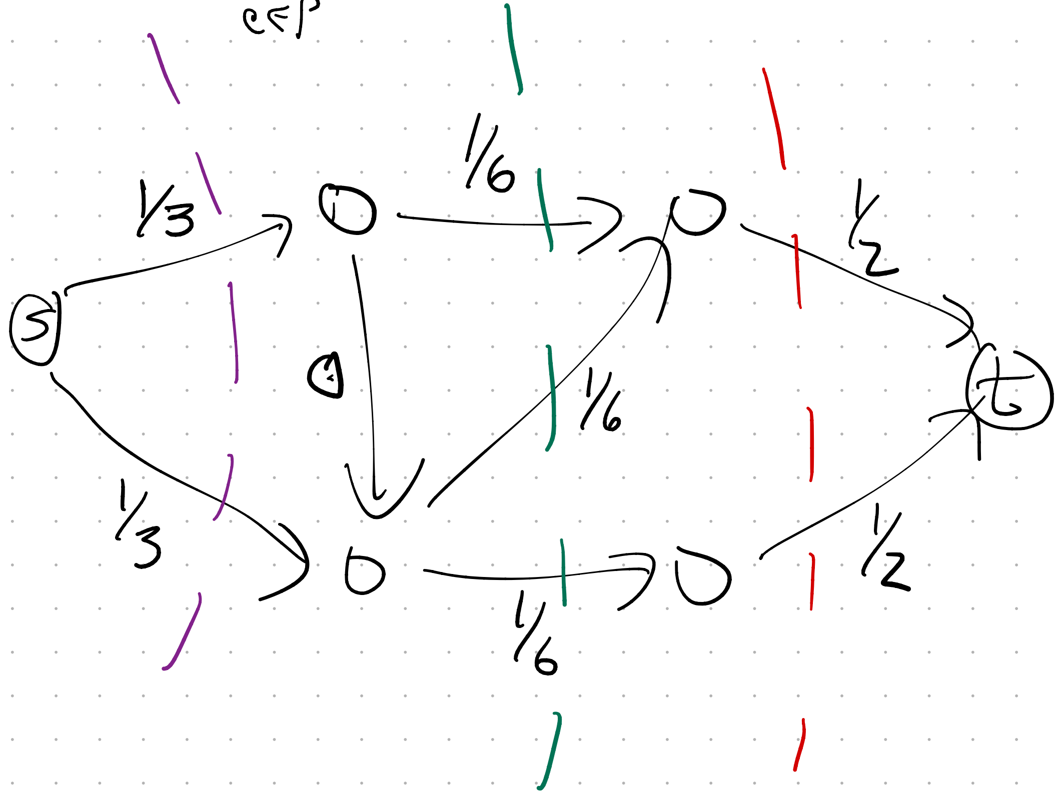
where $s \in A$, $t \in B$

s-t cut set = superset of exact s-t cut set

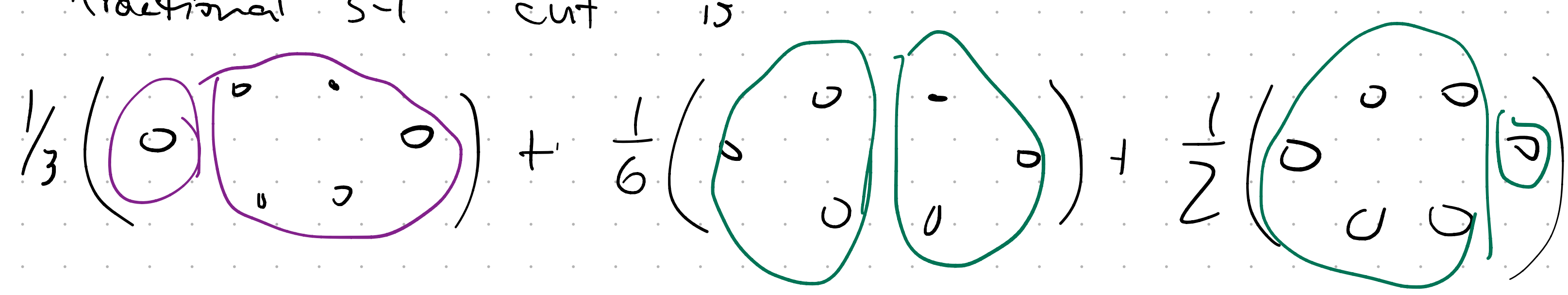
fractional s-t cut set $= x \in \mathbb{R}^E$ s.t.

$x_e \geq 0 \quad \forall e,$

$\sum_{e \in P} x_e \geq 1 \quad \forall P \in \mathcal{P}(s,t)$



This fractional s-t cut is



LEMMA (last time): Whenever $\vec{x} \in \mathbb{R}^E$ is a fractional s-t cut set, there is a convex combination of exact s-t cut sets, x', such that $x'_e \leq x_e \ \forall e$.

This shows...

<u>Cor.</u> For any vector $\vec{c} \in \mathbb{R}^E_{\geq 0}$ ("capacities")

$$\min \sum_e c(e) x_e$$
$$\text{s.t.} \sum_{e \in P} x_e \geq 1 \quad \forall P \in \mathcal{P}(s,t)$$
$$x_e \geq 0$$

the minimum is attained at a $\{0,1\}^E$ vector corresponding to an <u>exact</u> s-t cut.

<u>Proof.</u> Suppose $\vec{x} \in \mathbb{R}^E$ is a vector that attains the minimum value.

Using Lemma write $\vec{x} \geq \vec{x}'$ and

$$x' = \sum_{i=1}^{M} w_i \, x_{cut(A_i, B_i)}$$

$\{0,1\}$ vector corresponding to s-t cut (A_i, B_i).

where $w_i \geq 0 \ \forall i$, $\sum_i w_i = 1$.

Each $\vec{x}_{cut(A_i, B_i)}$ is feasible for the LP above. By def'n of x

$$\forall i \quad \sum_e c(e) x_{cut(A_i,B_i),e} \geq \sum_e c(e) x_e.$$

$$\sum_e \sum_i w_i \, c(e) \, x_{cut(A_i,B_i),e} \geq \sum_e c(e) x_e$$

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$$\sum_e c(e)\, x'_e \geq \sum_e c(e)\, x_e$$

On the other hand

$$\forall e \quad x_e \geq x'_e \quad \text{and} \quad c(e) \geq 0$$

$$\Rightarrow \quad c(e)\, x_e \geq c(e)\, x'_e$$

$$\sum_e c(e)\, x_e \geq \sum_e c(e)\, x'_e$$

If any $x_{cut(A_i, B_i)}$ had greater obj. value than x, that inequality would have been strict and $OBJ(x') = OBJ(x)$ would not hold.

$\therefore$ <u>each</u> $x_{cut(A_i, B_i)}$ attains the LP min.

Strong duality:

$$\min \sum c(e)\, x_e = c^T x \qquad c = \begin{bmatrix} c(e_1) \\ c(e_2) \\ \vdots \\ c(e_m) \end{bmatrix}$$

$$\text{s.t.} \quad \sum_{e \in P} x_e \geq 1 \quad \forall P \in \mathcal{P}(s,t) = Ax \geq \vec{1}$$

$$x_e \geq 0 \quad \forall e \in E$$

$$A = \overbrace{\begin{array}{c} \\ p \\ \\ \end{array}\begin{bmatrix} & \vdots & \\ \cdots & 1 \text{ if } e \in p & \cdots \\ & \vdots & \\ & e & \end{bmatrix}}^{\text{edges}} \Big\} \text{paths}$$

$$\max \sum_{p \in \mathcal{P}(s,t)} y_p$$

$$\text{s.t.} \quad \sum_{p:\, e \in p} y_p \leq c(e) \quad \forall e \in E$$

$$y_p \geq 0 \quad \forall p \in \mathcal{P}(s,t)$$

This is a path-packing problem called "maximum flow."

strong duality

$$\max(\text{flow LP}) = \min(\text{frac st cut set})$$

Corollary of Monday's lemma

$$= \min(\text{capacity of exact st cut set})$$

Flows are usually represented more succinctly as functions mapping edges (u,v) to numbers, $f(u,v)$.

For directed graph $G = (V,E)$, let

$$\overleftrightarrow{E} = E \cup \{(v,u) \mid (u,v) \in E\}.$$

Def. An st flow is a function $f: \overleftrightarrow{E} \to \mathbb{R}$ satisfying

- skew symmetry $\quad f(v,u) = -f(u,v)$

- flow conservation

$$\forall u \neq s,t \qquad \sum_v f(u,v) = 0$$

A flow is feasible for capacity function $c : \overleftrightarrow{E} \to \mathbb{R}$ if $f(u,v) \leq c(u,v) \quad \forall (u,v) \in \overleftrightarrow{E}$.

The value of a flow f is

$$val(f) = \sum_v f(s,v).$$

The maximum flow problem is: given $G=(V,E)$ and $c : \overleftrightarrow{E} \to \mathbb{R}_{\geq 0}$ find a feasible flow of maximum value.

Relation to $\max\left\{ \sum y_p \;\middle|\; \sum_{p:\, e \in P} y_p \leq c(e) \;\forall e,\; y \geq 0 \right\}$.

$$\left\{ \begin{matrix} \text{feasible } (y_p) \\ \text{vectors} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{feasible s-t} \\ \text{flows} \end{matrix} \right\}$$

$y \longmapsto f$ def'd by

* not surjective!

$$f(u,v) = \sum_{P:\,(u,v) \in P} y_p - \sum_{P:\,(v,u) \in P} y_p.$$

$$\sum_v f(u,v) = 0 \quad \text{when} \quad u \notin \{s,t\}.$$

$\quad\; \Vert$

$$\sum_v \sum_{P:\,(u,v) \in P} y_p - \sum_v \sum_{P:\,(v,u) \in P} y_p$$