$\boxed{\text{7 Oct 2024}}$ Cuts and Flows

Announcements.

(i) Problem Set 3 to be released Friday, 10/11, due 10/25.

(ii) Midterm is take-home, 48 hour limit.

You choose which date to start, in the range Oct 29 — Nov 1.

Email {Prof K, August, Linda} if you need a different set of dates.

$G = (V, E)$ directed graph, $n = |V|$, $m = |E| \geq n-1$.

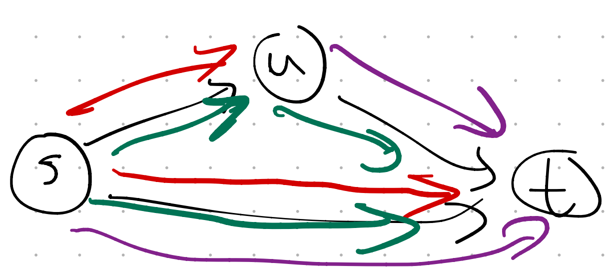
Two special vertices $s, t \in V$ called "source" and "sink".

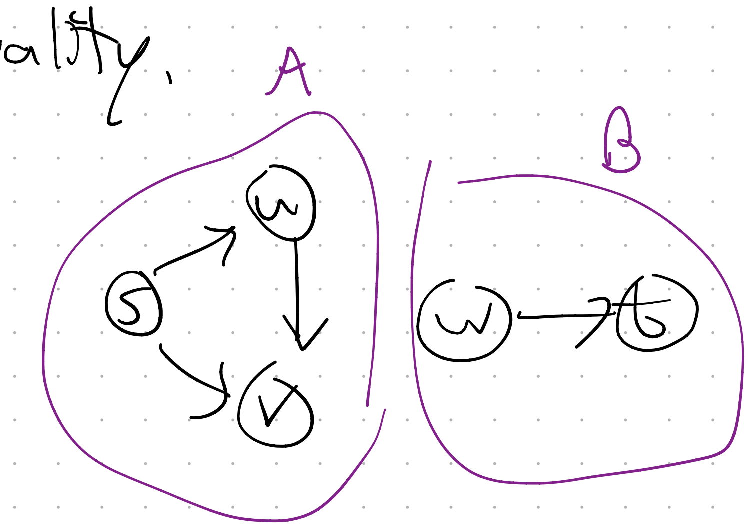
Def. An s-t cut is a partition $V = A \cup B$ such that $s \in A$, $t \in B$.

An edge set $\mathcal{C} \subseteq E$ will be called an <u>s-t cut set</u> (Bobby's nonstandard term) if

$$\exists \text{ s-t cut } A, B \text{ s.t. } \mathcal{C} \supseteq \{(u,v) \mid u \in A,\ v \in B\}.$$

$\mathcal{C}$ is an <u>exact</u> s-t cut set of the above $\supseteq$ is an equality.





<u>Lemma.</u> $\mathcal{C}$ is an s-t cut set if and only if it intersects the edge set of every path from s to t. (Notation $\mathcal{P}(s,t) = \{\text{s-t paths}\}$.)

Proof. Suppose A, B is an s-t cut and $\mathcal{C} = \{(u,v) \mid u \in A, v \in B\}$. Then every s-t path P intersects $\mathcal{C}$ because the first edge $(u,v) \in E(P)$ with $v \in B$, must satisfy $u \in A$, $v \in B$ then $(u,v) \in \mathcal{C}$.

Conversely, if $\mathcal{C}$ intersects every s-t path let $A = \{$vertices reachable from s using a path in complement of $\mathcal{C}\}$

$$B = V \setminus A.$$

By assumption $t \in B$ whereas $s \in A$.

Claim $\forall (u,v) \in E$ with $u \in A$, $v \in B$, we must have $(u,v) \in \mathcal{C}$.

Since $u \in A$, $\exists$ path P_u from s to u, $E(P_u) \cap \mathcal{C} = \emptyset$.

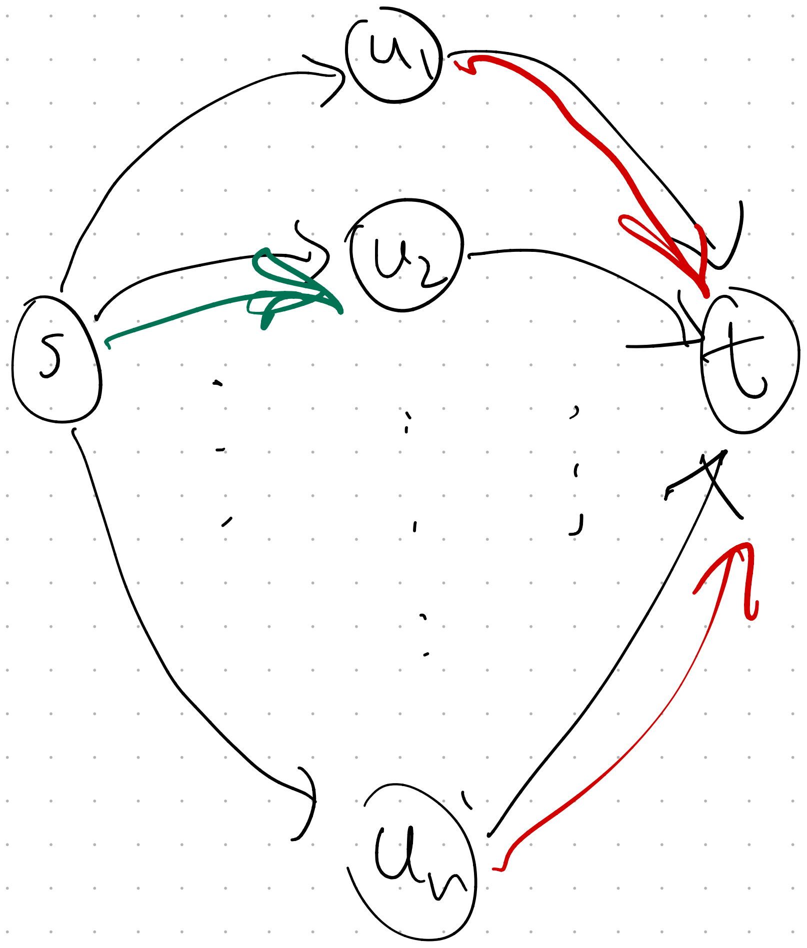
"append"

Let $P_v = P_u + (u,v)$.

Since $v \notin A$, P_v intersects $\mathcal{C}$.

As P_u is disjoint from $\mathcal{C}$, $(u,v) \in \mathcal{C}$.

Ex. A graph with 2^n minimal s-t cut sets and $O(n)$ vertices...



2^n ways to chose a minimal s-t cut set in this graph.

Def A fractional s-t cut set is a vector $\vec{x} \in \mathbb{R}^E$ that satisfies

$$\forall P \in \mathcal{P}(s,t) \quad \sum_{e \in P} x_e \geq 1.$$

$$\forall e \in E \qquad x_e \geq 0$$

Note $\vec{x} \in \{0,1\}^E$ satisfying the above constraints are in 1:1 correspondence with s-t cut sets.

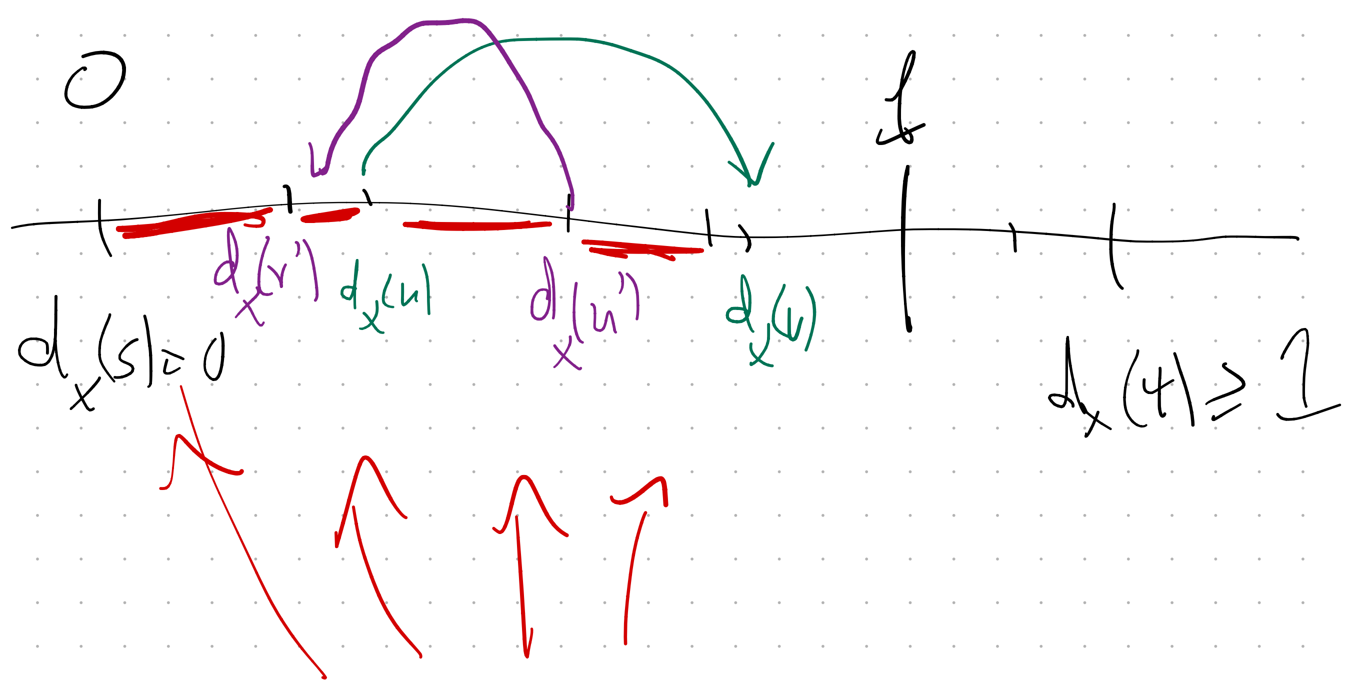
Lemma: (Like Birkhoff-vN but for cuts)
For every fractional s-t cut set, X, $\exists$ convex combination of s-t cut sets, X', s.t. $x \geq x'$.

Proof. Suppose x is a fractional s-t cut set. For $u \in V$ let
$$d_x(u) = \min_{P \in \mathcal{P}(s,u)} \left\{ \sum_{e \in P} x_e \right\}.$$

Note $d_x(t) \geq 1$ because x is a fractional s-t cut set.

For $0 < \theta < 1$ let
$$A_\theta = \{ u \mid d_x(u) \leq \theta \}$$
$$B_\theta = \{ u \mid d_x(u) > \theta \}$$
$$\mathcal{C}_\theta = \{ (u,v) \mid u \in A_\theta,\ v \in B_\theta \}.$$

Let $x'_e := \text{measure}(\{ \theta \mid e \in \mathcal{C}_\theta \})$.



to each red interval corresponds a cut set that equals C_θ when θ lands in that interval.

$x' =$ weighted average of those cut sets, weighted by interval lengths.

Exercise. $x \succeq_C x'$.