4 Oct 2024

Ellipsoid Algorithm (Part 2)

Today: An algorithm for solving:

Find $x \in P \subseteq \mathbb{R}^n$, P convex

or conclude (correctly) P is empty

Given:

- a separation oracle for P
- a promise if $P \neq \emptyset$ then
$$\mathrm{vol}\big(P \cap \underbrace{B_2(R)}_{\text{Euclidean ball, radius } R}\big) \geq \delta.$$

Running time and # separation oracle calls will be $\mathrm{poly}(n, \log(R), \log(1/\delta))$.

Description of algorithm:

1. Let $\mathcal{E}_0 = B_2(R)$ // $\mathrm{vol}(P \cap \mathcal{E}_0) \geq \delta$.
2. For $i = 0, 1, 2, \ldots$:

   Let x_i = center of $\mathcal{E}_i$

   Query separation oracle at x_i,

   if $x_i \in P$:

   halt and output x_i

   else:

   oracle returns a_i such that

   $a_i^T(x - x_i) > 0 \quad \forall x \in P$

let $\mathcal{E}_{i+1}$ = an ellipsoid of volume $\leq \left(1+\frac{1}{2n}\right)^{-1} \mathrm{vol}(\mathcal{E}_i)$ containing $\mathcal{E}_i \cap \{x \mid a_i^T(x-x_i) \geq 0\}$

if $\mathrm{vol}(\mathcal{E}_{i+1}) < \delta$:

halt and output "$P = \emptyset$".

Loop Invariant: $\mathcal{E}_i$ contains $P \cap \mathcal{E}_0$.

True when $i = 0$.

Induction step:

$$\mathcal{E}_{i+1} \supseteq \mathcal{E}_i \cap \{x \mid a_i^T(x-x_i) \geq 0\}$$

$\mathcal{E}_i \supseteq P \cap \mathcal{E}_0$ $\{x \mid a_i^T(x-x_i) \geq 0\} \supseteq P$, $P \cap \mathcal{E}_0 \subseteq P$

Induct Hypoth   Sep Oracle

$$\therefore \quad P \cap \mathcal{E}_0 \subseteq \mathcal{E}_{i+1}.$$

Bound on # iterations:

After t iterations

$$\delta \leq \mathrm{vol}(\mathcal{E}_t) \leq \left(1+\frac{1}{2n}\right)^{-t} \mathrm{vol}(\mathcal{E}_0)$$

$$\left(1+\frac{1}{2n}\right)^{t} \leq \delta^{-1} \mathrm{vol}(\mathcal{E}_0)$$

$$t \leq \frac{\ln(\mathrm{vol}(\mathcal{E}_0)) - \ln(\delta)}{\ln\left(1 + \frac{1}{2n}\right)}$$

$$\leq O(n) \cdot \left[n \ln(2R) - \ln(\delta) \right].$$

Why does $\exists\ \mathcal{E}_{i+1} \supseteq \mathcal{E}_i \cap \mathcal{H}_i$

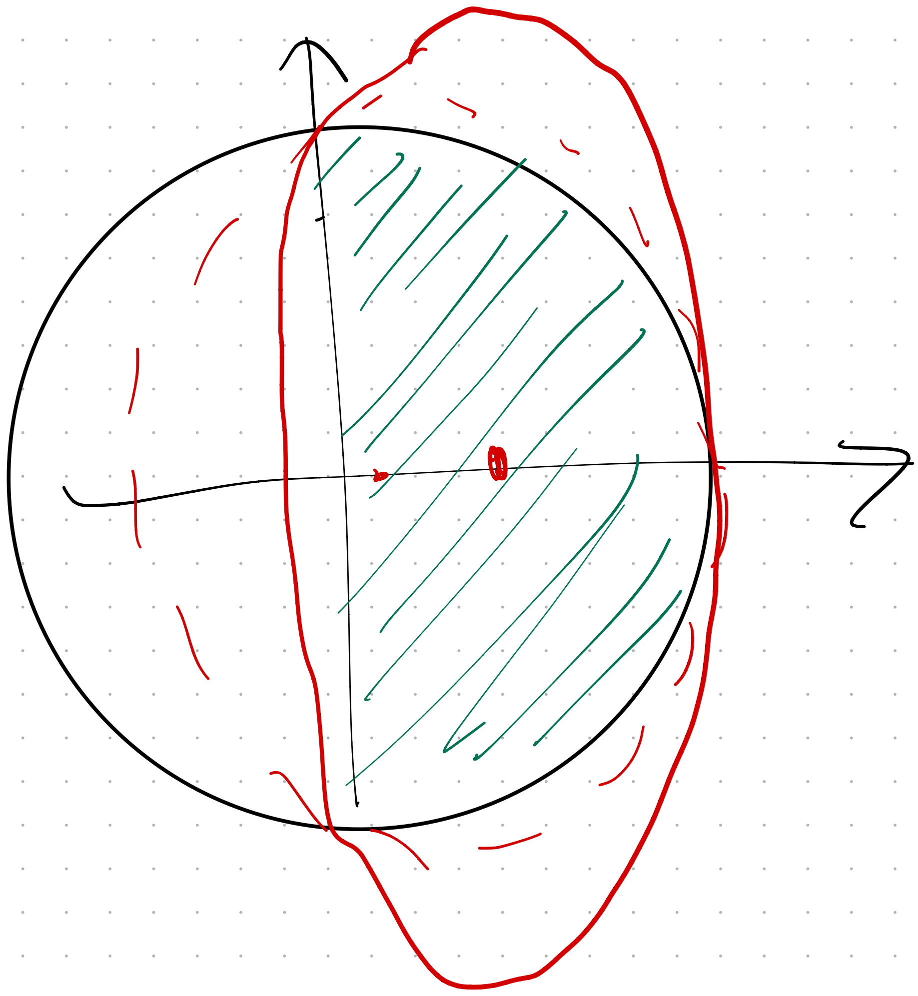
s.t. $\mathrm{vol}(\mathcal{E}_{i+1}) \leq \left(1 + \frac{1}{2n}\right)^{-1} \mathrm{vol}(\mathcal{E}_i)$?

Up to invertible affine transformations, it suffices to answer this when

$$\mathcal{E}_i = B_2^n(1) = \{ x \mid \|x\|_2^2 \leq 1 \}$$

and

$$\mathcal{H}_i = \{ x \mid x_1 \geq 0 \}.$$



For ellipsoid centered at

$$\begin{bmatrix} \frac{\delta}{1+\delta} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

then $\mathcal{E}_{i+1} = \left\{ x \,\middle|\, (1+\delta)^2 \left(x_1 - \frac{\delta}{1+\delta}\right)^2 + (1-\delta^2) \sum_{i=2}^{n} x_i^2 \leq 1 \right\}$

Contains $\mathcal{E}_i$.

$$\mathrm{Vol}(\mathcal{E}_{i+1}) = (1+\delta)^{-1} \cdot (1-\delta^2)^{-\frac{(n-1)}{2}} \cdot \mathrm{Vol}(\mathcal{E}_i)$$

Set $\delta = \frac{1}{n}$

$$\Longrightarrow \mathrm{vol}(\mathcal{E}_{i+1}) \leq \left(1 + \frac{1}{2n}\right)^{-1} \mathrm{vol}(\mathcal{E}_i)$$