2 Oct 2024 The Ellipsoid Algorithm

Def. A <u>separation</u> <u>oracle</u> for a polyhedron $P \subseteq \mathbb{R}^n$ is a function that takes $x \in \mathbb{R}^n$ and outputs either

- YES (only when $x \in P$)
- a violated constraint, meaning a pair (a, b) $a \in \mathbb{R}^n, b \in \mathbb{R}$
  s.t. $a^T x' \geq b$ is satisfied $\forall x' \in P$
  but $a^T x < b$.

The bit complexity of the oracle is B if the pair (a,b) is always in

$$(\mathbb{Z}^n \times \mathbb{Z}) \cap \left(-2^{B-1}, 2^{B-1}\right]^{n+1}.$$

i.e. B is # of bits in the output.

Ellipsoid alg solves LP with n variables in poly(n, B) calls to a separation oracle with bit complexity B.
(Plus poly(n,B) additional computation.)

<Ex> Suppose $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$

and $P = \{ x \mid Ax \geq b \}$.

Separation oracle implementation:

for $i \in [m]$:

  Let $a_i^T = i^{th}$ row of A,

  compute $a_i^T x$

  if $a_i^T x < b_i$ return (a_i, b_i)

endfor

return YES.

Bit complexity is $\lceil \max\{ \log_2(|a_{ij}|), \log_2(|b_i|) \} \rceil$

<Ex/>

Example: Network monitoring.

Input: Directed graph $G = (V, E)$

Src-dest pairs $\{ (s_i, t_i) \mid i = 1, \ldots, k \}$

A drug trafficking ring is trying to smuggle from s_i to t_i along a path in G.

An enforcement agency knows

G, $\{(s_i, t_i) \mid i = 1, \dots, k\}$

but doesn't know i nor path.

Goal: if possible, try to find a prob distrib (x_e) on edges of G, such that

$$\forall i \in [k] \quad \forall \text{ path } P \text{ from } s_i \text{ to } t_i$$

$$\sum_{e \in P} x_e \geq 0.1$$

Reducing LP optimization to LP search. (Find a feasible vector if one exists.)

<u>Given:</u>

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

Solve the following search problem over pairs of vectors $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$.

$$\left\{ Ax \leq b,\ A^T y \geq c,\ c^T x = b^T y,\ x \geq 0, y \geq 0 \right\}$$

<u>Def.</u> An ellipsoid in $\mathbb{R}^n$ is the image of the unit ball

$$\mathbb{B} = \{ x \mid \|x\|_2^2 \leq 1 \}$$

under an invertible affine transformation $x \mapsto y = x_0 + Tx$

The ellipsoid alg. is best described in terms of solving a search problem over polyhedron $\mathbb{P}$ given $0 < \gamma$, $R < \infty$ such that

- if $\mathbb{P} \neq \emptyset$ then $\mathbb{P} \cap (R \cdot \mathbb{B}) \neq \emptyset$.
- if $\mathbb{P} \neq 0$ then $\mathrm{vol}(\mathbb{P}) \geq \gamma$.

Running time will be $\mathrm{poly}(n, B, \log_2(R), \log_2(\frac{1}{\gamma}))$.