30 Sep 2024

Finishing Strong Duality
Running Times for LP

Announcements. (1) PSet 1 is graded

Regrades avail. until 10/7

(2) PSet 2: meaning of point allocation

An LP in equational form is sometimes written as

$$\max \ c^T x$$
$$\text{s.t. } Ax = b$$
$$x \geq 0$$

(most succinct)

or as

"slack variables"

$$\max \ c_1 x_1 + \dots + c_n x_n$$
$$\text{s.t. } a_{11}x_1 + \dots + a_{1n}x_n + w_1 = b_1$$
$$a_{21}x_1 + \dots + a_{2n}x_n + w_2 = b_2$$
$$\vdots$$
$$a_{m1}x_1 + \dots + a_{mn}x_n + w_m = b_m$$
$$x, w \geq 0$$

To finish up the duality proof, must be more careful about distinguishing original variables of the (standard form) LP and slack variables.

Equational form will be written as

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax + w = b \\ & x, w \geq 0 \end{aligned}$$

We finished the simplex algorithm in one of 3 states.

(a) infeasible

(b) feasible but unbounded
($c^T x$ may be arbitrarily large for x satisfying $Ax \leq b, x \geq 0$.)

(c) simplex terminates because obj func has been rewritten in an equivalent form with only non-positive coefficients on the variables.

That means:

$$\underbrace{c^T x = k - \eta^T x - \xi^T w}_{\text{valid for all } x, w \text{ satisfying } Ax + w = b.}$$

$$\eta \geq 0, \quad \xi \geq 0$$

$$(c+\eta)^T x + \xi^T w = k \quad \text{when} \quad Ax + w = b$$

If $\quad Ax' + w' = b = Ax + w \quad$ then

$$(c+\eta)^T x' + \xi^T w' = k = (c+\eta)^T x + \xi^T w.$$

$\iff$ If $\quad A(x'-x) + (w'-w) = 0$

then $\quad (c+\eta)^T (x'-x) + \xi^T (w'-w) = 0$

$m \times m$ identity

$\iff$ If $\quad \begin{bmatrix} A \,|\, \mathbb{1} \end{bmatrix} \begin{bmatrix} x'-x \\ w'-w \end{bmatrix} = 0$

then $\quad \begin{bmatrix} (c+\eta)^T \,|\, \xi^T \end{bmatrix} \begin{bmatrix} x'-x \\ w'-w \end{bmatrix} = 0$

$\Rightarrow \quad \begin{bmatrix} (c+\eta)^T \,|\, \xi^T \end{bmatrix}$ is in the rowspace of $\begin{bmatrix} A \,|\, \mathbb{1} \end{bmatrix}$.

$\Rightarrow \begin{bmatrix} c+\eta \\ \xi \end{bmatrix}$ in column space of $\begin{bmatrix} A^T \\ \mathbb{1} \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} c+\eta \\ \xi \end{bmatrix} = \begin{bmatrix} A^T \\ \mathbb{1} \end{bmatrix} \xi$$

$$\Rightarrow A^T \xi = c+\eta \Rightarrow \xi^T A = (c+\eta)^T$$

$$\Rightarrow A^T \xi \succeq c \quad \text{and} \quad \xi \succeq 0$$

Recall dual LP:

$$\begin{aligned} \min \quad & b^T y \\ \text{s.t.} \quad & A^T y \geq c \\ & y \succeq 0 \end{aligned}$$

Recall: $Ax + w = b \Rightarrow (c+\eta)^T x + \xi^T w = k.$

If $Ax + w = b$ then

$$\xi^T A x + \xi^T w = \xi^T b$$

$$\Longrightarrow \quad (c+\eta)^T x + \xi^T w = \xi^T b = b^T \xi$$

$$\Longrightarrow \quad K = b^T \xi .$$

Conclusion: Simplex algorithm

finds x satisfying PRIMAL

$y = \xi$ satisfying DUAL

s.t. PRIMAL OBJ (x)

$=$ DUAL OBJ (y)

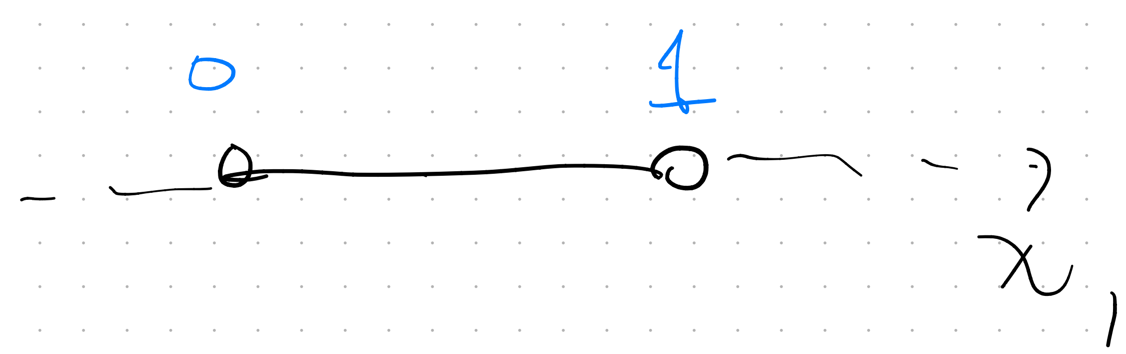
$\therefore$ both are opt for their respective LP's.

Klee-Minty cube: for $0 < \delta \ll 1$

$$\max \quad x_n$$

$$\text{s.t.} \quad 0 \le x_1 \le 1$$
$$\delta x_1 \le x_2 \le 1 - \delta x_1$$
$$\delta x_2 \le x_3 \le 1 - \delta x_2$$
$$\vdots$$
$$\delta x_{n-1} \le x_n \le 1 - \delta x_{n-1}$$

n=1:



n=2: