27 Sep 2024

The Simplex Method

Recall. We are working with LP in equational form.

$$\max \quad c^T x$$
$$\text{s.t.} \quad Ax = b$$
$$x \geq 0$$

E.g.

$$\max \quad 2x_1 + 3x_2$$
$$\text{s.t.} \quad x_1 + x_2 + w_1 = 8$$
$$2x_1 + x_2 + w_2 = 12$$
$$x_1 + 2x_2 + w_3 = 14$$
$$x_1, x_2, w_1, w_2, w_3 \geq 0$$

Assume: (WLOG.... to be justified below)

We know a feasible starting point where some variables are set to zero and the rest are determined by linear eq's.

Solving LP using simplex method really consists of solving 2 LP's with different objectives.

First LP: find minimum slack that makes the actual LP you want to solve feasible.

Second LP: Solve the actual LP

First LP: min w_0

$$
\begin{aligned}
\text{s.t.}\quad & x_1 + x_2 + w_1 - w_0 = 8 \\
& 2x_1 + x_2 + w_2 - w_0 = 12 \\
& x_1 + 2x_2 + w_3 - w_0 = 14 \\
& x_1, \ldots, \geq 0
\end{aligned}
$$

If OPT > 0, actual LP is infeasible $\Longrightarrow$ HALT.

Else OPT $= 0$, and solving the LP above using simplex gives us a vertex of the actual LP's feasible set.
Start solving the actual LP from that vertex.

Second LP:

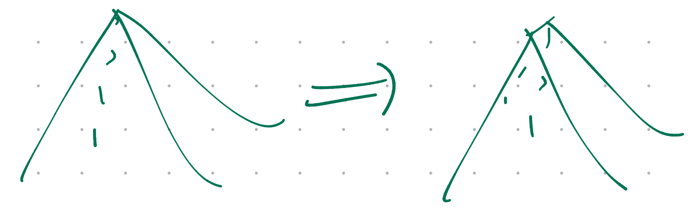
$$
\begin{aligned}
\max\quad & c^T x \\
\text{s.t.}\quad & Ax = b \\
& x \geq 0
\end{aligned}
$$

In each iteration, some variables ("non-basic variables") are set to zero and the others' ("basic variables") values are determined by the lin equations.

Furthermore, the objective function has been rewritten as an affine function of the non-basic variables. linear function plus constant.

"Pivot": pick one non-basic variable whose coefficient in the obj. is positive. If no such var exists, HALT.
Increase its value as much as possible, until some basic variable decreases to $\emptyset$.

At that point the basic var that reaches $\emptyset$ becomes non-basic and the pivot is complete.



$$\max \quad 2x_1 + 3x_2$$

$$\text{s.t.} \quad \begin{aligned} x_1 + x_2 + w_1 &= 8 \\ 2x_1 + x_2 + w_2 &= 12 \\ x_1 + 2x_2 + w_3 &= 14 \\ \vec{x}, \vec{w} &\geq 0 \end{aligned}$$

$w_1 = 8 - x_1 - x_2$

$\boxed{w_2 = 12 - 2x_1 - x_2}$ $\quad x_1 = 6$, $w_2 = 0$

$w_3 = 14 - x_1 - 2x_2$

$x_1 = 6 - \frac{1}{2}x_2 - \frac{1}{2}w_2$

$$\max \quad (12 - x_2 - w_2) + 3x_2 = 12 + 2x_2 - w_2$$

s.t. Same constraints as above. $\quad x_2 = 4$, $w_1 = 0$

$$6 - \tfrac{1}{2}x_2 - \tfrac{1}{2}w_2 + x_2 + w_1 = 8$$

$\boxed{w_1 = 2 - \frac{1}{2}x_2 + \frac{1}{2}w_2}$

$x_1 = 6 - \frac{1}{2}x_2 - \frac{1}{2}w_2$

$$6 - \tfrac{1}{2}x_2 - \tfrac{1}{2}w_2 + w_3 = 14$$

$w_3 = 8 + \frac{1}{2}x_2 + \frac{1}{2}w_2$

If the alg. terminates at a vertex where the objective function has a non-positive coefficient on every non-basic variable, i.e.

$$\max \quad k - \eta^T x \qquad (\eta \geq 0)$$

$$\text{s.t.} \quad Ax = b \qquad \forall i \text{ s.t. } \eta_i > 0,\ x_i = 0.$$

$$x \geq 0$$

where $k \in \mathbb{R}$, $k - \eta^T x = c^T x$ on the solution set of $Ax = b$.

At the solution we found $OBJ = k$.

At ~~any~~ feasible point $OBJ = k - \eta^T x$

$\leq k$.

So the solution we found maximizes OBJ.

Furthermore, $\left(\forall x \quad Ax = b \Rightarrow k - \eta^T x = c^T x \right)$

means that $(c+\eta)^T x$ is constant on the nullspace of A.