$\boxed{\text{25 Sep 2024}}$ Duality & Simplex Method

Recall: For a LP in standard form

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

there is a dual

$$\begin{aligned} \min \quad & b^T y \\ \text{s.t.} \quad & A^T y \geq c \\ & y \geq 0 \end{aligned}$$

WEAK DUALITY says OPT(DUAL) $\geq$ OPT(PRIMAL) and has 3-line algebraic proof.

STRONG DUALITY says OPT(DUAL) = OPT(PRIMAL) and has a proof we'll see Fri or Mon.

Running example

$$\begin{aligned} \max \quad & 2x_1 + 3x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 8 \\ & 2x_1 + x_2 \leq 12 \\ & x_1 + 2x_2 \leq 14 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$$\begin{aligned} \max \quad & \begin{bmatrix} 2 \\ 3 \end{bmatrix}^T \vec{x} \\ \text{s.t.} \quad & \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 2 \end{bmatrix} \vec{x} \leq \begin{bmatrix} 8 \\ 12 \\ 14 \end{bmatrix} \\ & \vec{x} \geq 0 \end{aligned}$$

OPT: $x_1 = 2$ $x_2 = 6$

$$\min \quad 8y_1 + 12y_2 + 14y_3 \qquad \min \begin{pmatrix} 8 \\ 12 \\ 14 \end{pmatrix}^T \vec{y}$$

$$\text{s.t.} \quad y_1 + 2y_2 + y_3 \geq 2 \qquad \text{s.t.} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \vec{y} \geq \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$y_1 + y_2 + 2y_3 \geq 3 \qquad \vec{y} \geq 0$$

$$y_1, y_2, y_3 \geq 0$$

OPT: $y_1 = 1,\ y_2 = 0,\ y_3 = 1$

"Perturbative" interpretation of duals...

$$\max \quad 2x_1 + 3x_2$$

$$\text{s.t.} \quad x_1 + x_2 \leq 8 + \varepsilon \qquad [y_1]$$

$$2x_1 + x_2 \leq 12 \qquad [y_2]$$

$$x_1 + 2x_2 \leq 14 \qquad [y_3]$$

$$x_1, x_2 \geq 0$$

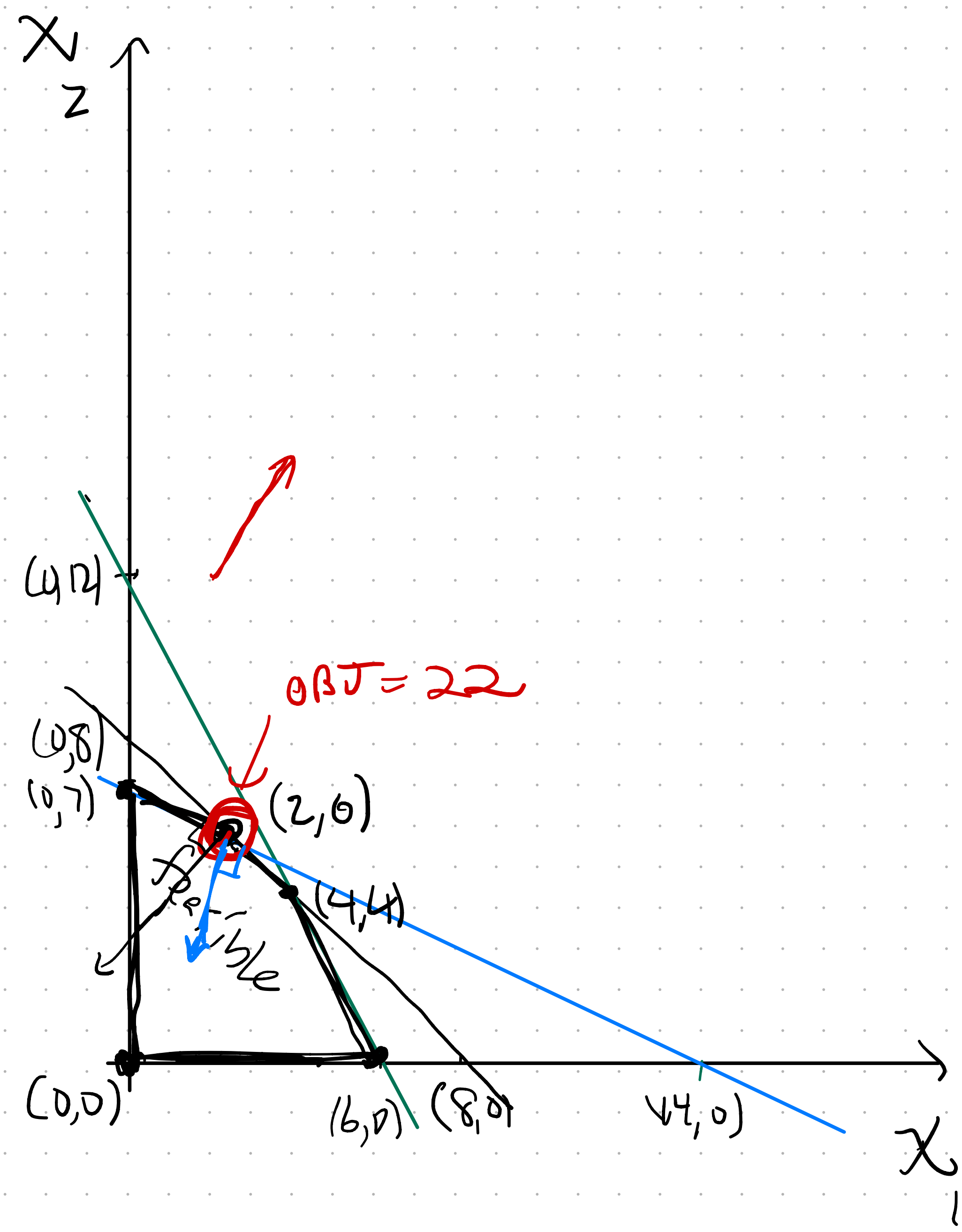
Coeff. of ε equals $\underline{1}$

OPT: $x_1 = 2 + 2\varepsilon \quad x_2 = 6 - \varepsilon$

... corresponding to $y_1 = 1$ at dual opt.

OBJ. VAL: $2x_1 + 3x_2 = 4 + 4\varepsilon + 18 - 3\varepsilon = 22 + \varepsilon$

Physical interpretation in terms of force vectors.



Economic interpretation

$$\min \quad 8y_1 + 12y_2 + 14y_3$$

s.t.

$$y_1 + 2y_2 + y_3 \geq 2$$

$$y_1 + y_2 + 2y_3 \geq 3$$

$$y_1, y_2, y_3 \geq 0$$

$$\max \quad 2x_1 + 3x_2$$

s.t.

$$x_1 + x_2 \leq 8$$

$$2x_1 + x_2 \leq 12$$

$$x_1 + 2x_2 \leq 14$$

$$x_1, x_2 \geq 0$$

Equational form of a linear program.

Modify an LP in standard form by adding "slack variables" $w_1, \dots, w_m$ s.t. inequalities become equations.

E.g.

$$\max \quad 2x_1 + 3x_2$$

$$\begin{aligned} \text{s.t.} \quad x_1 + x_2 + w_1 &= 8 \\ 2x_1 + x_2 + w_2 &= 12 \\ x_1 + 2x_2 + w_3 &= 14 \\ \vec{x}, \vec{w} &\geq 0 \end{aligned}$$