23 Sep 2024

Linear Programming

Running example

$$\begin{aligned} \max \quad & 2x_1 + 3x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 8 \\ & 2x_1 + x_2 \leq 12 \\ & x_1 + 2x_2 \leq 14 \\ & x_1, x_2 \geq 0 \end{aligned}$$

3 problems

i. LP feasibility: given a system of linear inequalities in n variables, does it have a solution in $\mathbb{R}^n$?

ii. LP optimization: given a system of linear inequalities whose solution set is called the "feasible region", what is the maximum or minimum value attained by some given linear objective function on the feasible region?

iii. LP search: given an LP feasibility problem, find a point in the feasible region if one exists.

Or, given an LP optimization problem, find a feasible point where the obj function attains its max or min.

Ways of looking at a LP...

1. Algebraically:

$$\max \quad c_1x_1 + c_2x_2 + \dots + c_nx_n$$

$$\text{s.t.} \quad a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \leq b_1$$

$$\vdots \qquad\qquad \vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \leq b_m$$

$$\Downarrow$$

coefficient vector

$$\max \quad c^T x \quad \leftarrow \text{solve for } x \in \mathbb{R}^n$$

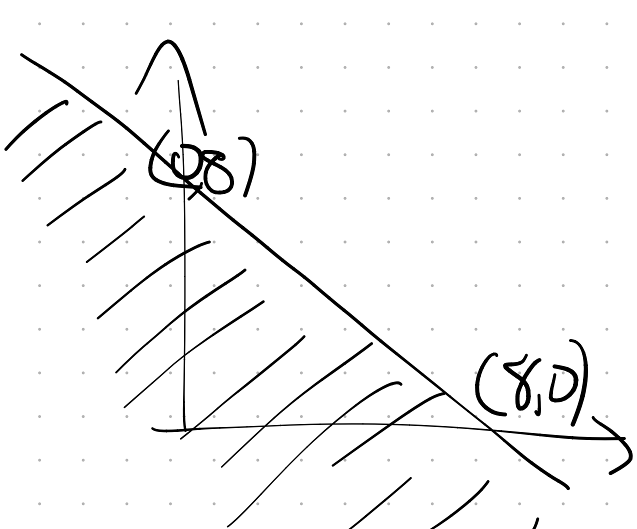
coordinate-wise $\leq$

$$\text{s.t.} \quad Ax \leq b \leftarrow \text{RHS vector}$$

constraint matrix

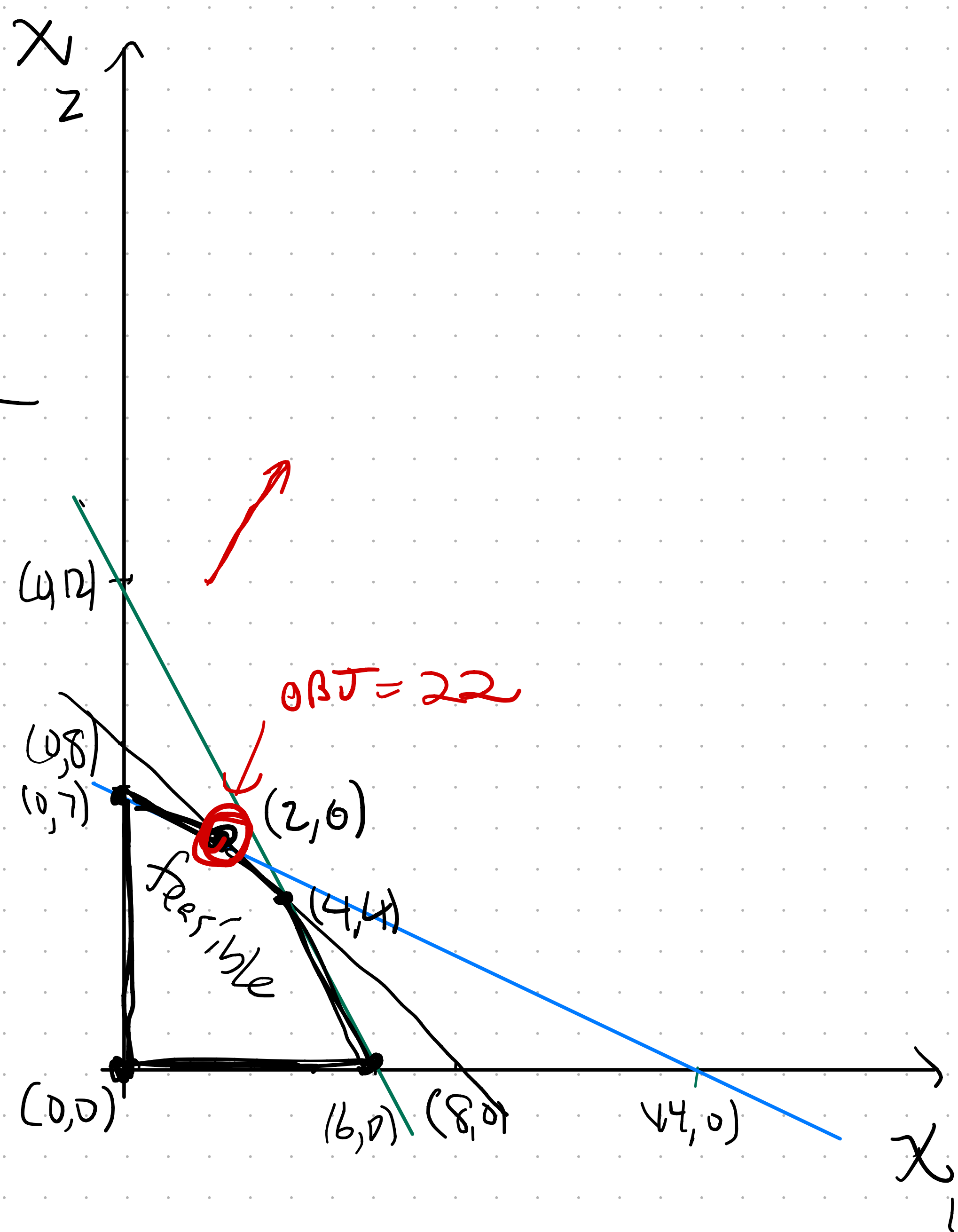
2. Geometrically: $a_{11}x_1 + \dots + a_{1n}x_n \leq b_1$ defines a "halfspace". The solution set consists of a hyperplane in $\mathbb{R}^n$ together with all the points on one side of it.

E.g. $x_1 + x_2 \leq 8$



The feasible region defined by $Ax \leq b$ is an intersection of finitely many halfspaces $\Rightarrow$ a polyhedron.

$$\begin{aligned} \max \quad & 2x_1 + 3x_2 \\ \text{s.t.} \quad & x_1 + x_2 \le 8 \\ & 2x_1 + x_2 \le 12 \\ & x_1 + 2x_2 \le 14 \\ & x_1, x_2 \ge 0 \end{aligned}$$



Brute force is exponential-time in higher dimensions. If you have n variables and m inequalities, what is an upper bound on # vertices of feasible region?

At every vertex there are (at least) n linear inequalities that are tight (hold with equality) and have linearly independent coefficient vectors.

These n linearly indep equations have a unique solution.

$\Rightarrow$ distinct vertices have distinct sets of tight $\le$

(# vertices) $\le$ (# of n-element subsets of the constraints)

$$= \binom{m}{n}$$

Standard form of an LP: constrain variables to be ≥ 0.

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \le b \\ & x \ge 0 \end{aligned} \qquad\qquad \begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \ge b \\ & x \ge 0 \end{aligned}$$

A LP in standard form has a <u>dual</u> which is also a LP.

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \le b \\ & x \ge 0 \end{aligned} \qquad\qquad \begin{aligned} \min \quad & b^T y \\ \text{s.t.} \quad & A^T y \ge c \\ & y \ge 0 \end{aligned}$$

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \ge b \\ & x \ge 0 \end{aligned} \qquad\qquad \begin{aligned} \max \quad & b^T y \\ \text{s.t.} \quad & A^T y \le c \\ & y \ge 0 \end{aligned}$$

If y is feasible for this problem and x is feasible for the primal then

$$b^T y \geq (Ax)^T y$$

$$= x^T A^T y$$

$$= x^T (A^T y)$$

$$\geq x^T c = c^T x$$

weak duality

This proves $\boxed{OPT(DUAL) \geq OPT(PRIMAL)}$

when DUAL is minimization

and PRIMAL is maximization.