$\boxed{\text{20 Sep 2024}}$ Lovász's Perfect Matching Decision Algorithm

Recall $G=(V,E)$ bipartite. $V = L \cup R$

$|L| = |R| = n$.

$$A_G = \text{matrix with} \quad a_{i,j} = \begin{cases} 1 & \text{if } \{u_i, v_j\} \in E \\ \emptyset & \text{if } \{u_i, v_j\} \notin E. \end{cases}$$

<u>Def.</u>

$$\text{per}(A) = \sum_{\substack{\text{permutations} \\ \sigma : [n] \to [n]}} \prod_{i=1}^{n} a_{i,\sigma(i)}$$ (Valiant: #P-hard)

$(-1)^{\#\text{ transpositions in } \sigma}$

$$\det(A) = \sum_{\substack{\text{permutations} \\ \sigma : [n] \to [n]}} \text{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}$$ (poly-time)

$\text{per}(A_G) = \#$ perfect matchings of G

$\det(A_G) \equiv \text{per}(A_G) \mod 2$

$\Longrightarrow$ In poly-time we can decide whether G has an even or odd # of perfect matchings.

Computing $n \times n$ determinant takes $O(n^\omega)$ arithmetic operations

where $\omega =$ "exponent of matrix multiplication"

$= \inf \{ \nu : \exists$ matrix mult alg running in time $O(n^\nu) \}$.

We know $\omega < 2.373\ldots$

Huge open question: $\omega \stackrel{?}{=} 2$.

Lovasz's Algorithm: Let B_G be the matrix obtained from A_G by substituting random numbers x_{ij} when $a_{ij} = 1$, $\emptyset$ when $a_{ij} = 0$.

Let $\mathbb{F}$ be any field (e.g. rational numbers) and $S \subseteq \mathbb{F}$ be a set of at least $2n^2$ distinct elements of $\mathbb{F}$.

$\{x_{ij}\}$ will be independent, uniformly random elements of S.

Algorithm: if $\det(B_G) \neq 0$ output YES

if $\det(B_G) = 0$ output PROBABLY NO.

Note. Runs in $O(n^\omega)$.

Correctness? Will show that if alg outputs YES G must have a perfect matching.

If G has a perfect matching then $\Pr(\text{output PROBABLY NO}) < \frac{1}{2}$.

i.e. false positive rate $= 0$, False negative rate $< \frac{1}{2}$.

$$\det(B_G) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \underbrace{\prod_{i=1}^{n} x_{i,\sigma(i)}}$$

This can only be $\neq 0$
if $\{(u_i, v_{\sigma(i)})\}$ is a perf matching.

Bounding the false negative rate...

<u>Schwartz-Zippel Lemma</u>: if $P(x_1, \ldots, x_m)$ is a non-zero m-variate polynomial with coefficients in a field $\mathbb{F}$, and

d = max exponent of any variable in any monomial of P

and if we sample $x_1, \ldots, x_m$ independently at random from distributions s.t.

$$\max_{y \in \mathbb{F}} \max_{i \in [m]} \Pr(x_i = y) \leq \delta,$$

then $\Pr\left(P(x_1, \ldots, x_m) = 0\right) \leq md\delta$.

In Lovasz algorithm $m = \#E(G) \leq n^2$,
$\delta = |S| < \frac{1}{2n^2}$
$d = 1$

$$\Pr\left(\det(B_G) = 0\right) \leq n^2 \cdot 1 \cdot \frac{1}{2n^2} < \frac{1}{2}$$

... as long as G has a perfect matching.

Proof of Schwartz-Zippel: Induction on m.

$m=0 \Longrightarrow P$ is a non-zero scalar

$\Pr(P=0) = 0 = md\delta$

$m \geq 1 \Longrightarrow$ Consider P as a polynomial in x_m whose coeffs are poly's in $x_1, \ldots, x_{m-1}$.

$$P(x_1, \ldots, x_m) = \sum_{i=0}^{d} Q_i(x_1, \ldots, x_{m-1}) \cdot x_m^i$$

$P \neq 0 \Longrightarrow$ at least one $Q_i \neq 0$.

How could P evaluate to 0?

Case 1. We sample $x_1, \ldots, x_{m-1}$ and find that $Q_i(x_1, \ldots, x_{m-1}) = 0 \quad \forall i$.

IND HYP $\Longrightarrow$ Probability $\leq (m-1)d\delta$.

Case 2. $Q_i(x_1, \ldots, x_{m-1}) \neq 0$ but P evaluates to zero.

At most d distinct x_m values cause $P(x_1, \ldots, x_m) = 0$.

Each of them has $\Pr(x_m = y) \leq \delta$.

$\Longrightarrow$ Probability $\leq d\delta$.

Sum up the 2 cases, QED.