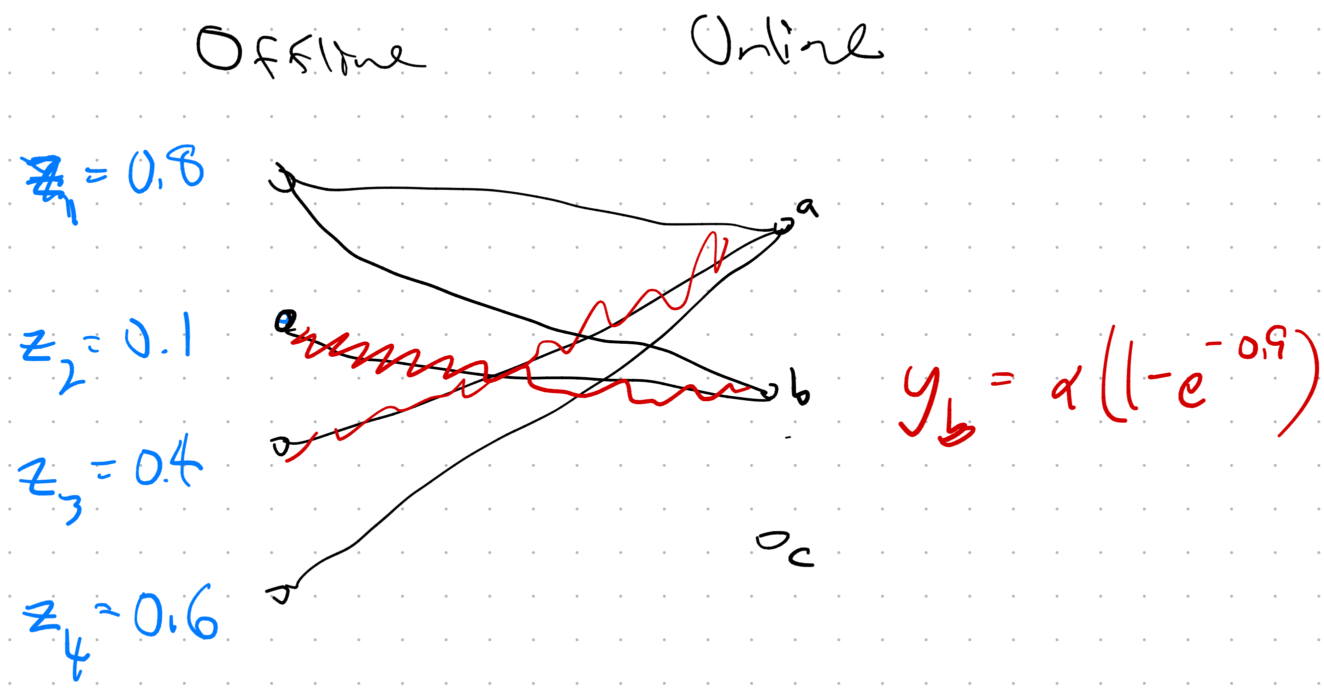
18 Sep 2024 — Finishing analysis of RANKING. Starting with algebraic matching algorithms

$\alpha = \frac{e}{e-1}$

$h(z) = e^{z-1}$



REMAINS TO SHOW: $E[\vec{y}]$ is feasible for DUAL.

$\forall$ edge $\{u,v\}$ we need $E[y_u] + E[y_v] \geq 1$.

Fix an edge $\{u,v\}$. Let $Z_{-u} = (z_{u'})_{u' \neq u}$

Define a quantity z^c depending on Z_{-u} by running RANKING on $G \setminus \{u\}$ using same random priorities Z_{-u}, see where v gets matched.

- unmatched $\Rightarrow$ $z^c = 1$
- matched to u' $\Rightarrow$ $z^c = z_{u'}$

We will show:

① $E[y_u | Z_{-u}] \geq \alpha \cdot \int_0^{z^c} h(t)\, dt$

② $E[y_v | Z_{-u}] \geq \alpha \cdot (1 - h(z^c))$

Summing: $E[y_u + y_v \mid Z_{-u}] \geq \alpha \left(1 - h(z^c) + \int_0^{z^c} h(t)\, dt\right) = \alpha \cdot \frac{1}{\alpha}$

To show ① ...

<u>Lemma:</u> When RANKING runs on G, if $z_u < z^c$ then u will definitely be matched.

<u>Proof.</u> Either u is matched before v arrives, or v picks u when it arrives provided that $z_u < z^c$.

To show ② ...

<u>Lemma.</u> When RANKING runs on G, if $z^c < 1$, v definitely gets matched to some w such that $z_w \le z^c$.

<u>Proof.</u> Induction on arrival time of vertices.

Induction hypothesis: for every $v' \in R$ the set of all free vertices when v' arrives in G is a superset of the free vertices when v' arrives in $G \setminus \{u\}$.

<u>Cor.</u> $$y_v = \alpha(1 - h(z_w)) \text{ if } (v,w) \text{ matched}$$
$$\geq \alpha(1 - h(z^c))$$

Matchings and Determinants

BIPARTITE PERFECT MATCHING DECISION PROBLEM.

Given bipartite $G = (V, E)$, $V = L \cup R$

where $|L| = |R| = n$

... does G have a perfect matching?

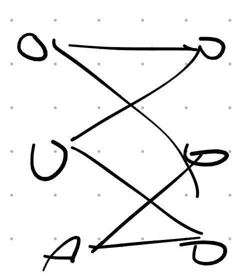
For a bipartite G with $|L| = |R| = n$ the bipartite adjacency matrix is defined by numbering the vertices $L = \{u_1, \ldots, u_n\}$

$R = \{v_1, \ldots, v_n\}$

and setting A_G to be the matrix with

$$a_{ij} = \begin{cases} 1 & \text{if } \{u_i, v_j\} \in E \\ 0 & \text{if } \{u_i, v_j\} \notin E. \end{cases}$$

Ex



$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$