16 Sep 2024 — Primal-dual analysis of RANKING.

Epilogue to ski rental lecture:

To transform online algorithms for PRIMAL LP into online randomized ski rental algorithm:

- On day t, when z increases to $z + \delta_t$,
  - BUY with probability δ_t
  - RENT with probability x_t
  - NEITHER with probability z (b/c you already own)

RANKING.

Recall $G = (V, E)$ $V = L \cup R$

Vertices in L are present at time $\emptyset$.

" " " R arrive at times $1, 2, \ldots, n$.

The set of neighbors of $v \in R$, denoted $N(v)$, is revealed when v arrives.

Set $\alpha = \frac{e}{e-1}$, $h(x) = e^{x-1}$.

RANKING

Initialize $y_u, y_v = 0$ $\forall u \in L, v \in R$.

Time $\emptyset$: sample independent, uniform random $z_u \in [0,1]$ for all $u \in L$.

Time $t > \emptyset$: vertex $v \in R$ arrives. If $\exists$ a free neighbor, match to the one with smallest z_u. Set $x_{uv} = 1$.

$y_u = \alpha \cdot h(z_u)$ $\qquad$ $y_v = \alpha \cdot (1 - h(z_u))$

Analysis. We will be augmenting the algorithm with book keeping steps that set dual variables y_u, y_v.

As with ski rental, these bookkeeping variables have an interpretation in terms of dual LP.

PRIMAL

$$\max \sum_{\{u,v\} \in E} x_{uv}$$

$$\text{s.t.} \quad \sum_{v \in N(u)} x_{uv} \le 1 \quad \forall u \in L$$

$$\sum_{u \in N(v)} x_{uv} \le 1 \quad \forall v \in R$$

$$x_{uv} \ge 0 \quad \forall \{u,v\} \in E$$

DUAL

$$\min \sum_{u \in L} y_u + \sum_{v \in R} y_v$$

$$\text{s.t.} \quad y_u + y_v \ge 1 \quad \forall \{u,v\} \in E$$

$$y_u, y_v \ge 0 \quad \forall u, v$$

Whenever RANKING updates x_{uv}'s and y_u's, y_v's,

$\sum x_{uv}$ increases by 1

$\sum y_u + \sum y_v$ increases by α.

At termination $\sum x_{uv} = |M|$ so

$$\sum_u y_u + \sum_v y_v = \alpha |M|.$$

The vector $\mathbb{E}\vec{x} = (\mathbb{E} x_{uv})_{\{u,v\} \in E}$ is feasible for PRIMAL LP.

If we can prove that $\mathbb{E}\vec{y} = (\mathbb{E} y_w)_{w \in V}$ is feasible for DUAL, then we know

$$\alpha \cdot \mathbb{E}|M| = \mathbb{E}\left[\sum y_u + \sum y_v\right] \geq \text{OPT (DUAL LP)}$$

$$\geq \text{OPT (PRIMAL LP)}$$

$$= \text{MAX MATCHING}$$

matching selected by RANKING.

To prove $\mathbb{E}y$ is dual feasible, recall we must show

$\forall \{u,v\} \in E \quad \mathbb{E}[y_u] + \mathbb{E}[y_v] \geq 1.$

$\forall u \in L, v \in R \quad \mathbb{E}[y_u] \geq 0$
$\mathbb{E}[y_v] \geq 0$

$y_v = \begin{cases} \alpha(1-h(z_u)) \\ 0 \text{ if unmatched} \end{cases}$

$y_u = \begin{cases} \alpha \cdot h(z_u) \text{ if matched} \\ 0 \text{ if unmatched} \end{cases}$

TRUE because $0 \leq h(z) \leq 1$ for $0 \leq z \leq 1$.

To reason about $\mathbb{E}y_u$, $\mathbb{E}y_v$ ....

Fix $Z_{-u} = (z_{u'})_{u' \in L \setminus \{u\}}$.

Aim to prove $\mathbb{E}[y_u + y_v \mid Z_{-u}] \geq 1 \quad \forall Z_{-u}$.

Define the <u>critical</u> <u>value</u> z^c to be a random variable, whose value depends on Z_{-u}, that equals $z_{u'}$, where u' is defined by running RANKING on $G \setminus \{u\}$ and finding the vertex to whom v is matched.

$z^c = 1$ if v remains unmatched.