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primal-dual analysis of online algorithms

LP relaxation of ski rental

$$\min \sum_{i=1}^{T} x_i + Bz$$

$$\text{s.t.} \quad x_i + z \geq 1 \quad \text{for } i=1,\dots,T$$

$$x_1,\dots,x_T, z \geq 0$$

Dual

$$\max \sum_{i=1}^{T} y_i$$

$$\text{s.t.} \quad y_i \leq 1 \quad \forall i$$

$$\sum_{i=1}^{T} y_i \leq B$$

$$y_i \geq 0 \quad \forall i$$

At time i we are going to set

$$y_i = \begin{cases} 1 & \text{if } i \leq B \\ 0 & \text{if } i > B \end{cases}$$

because we know that's guaranteed to be optimal for DUAL.

If we set a value of $x_i \leq x_{i-1}$

and set $z = 1 - x_i \geq 1 - x_{i-1}$

Then $\Delta \text{Primal} = x_i + B\Delta z = x_i + B(x_{i-1} - x_i)$

$$\Delta \text{Dual} = \begin{cases} 1 & \text{if } i \leq B \\ \emptyset & \text{if } i > B. \end{cases}$$

We want $\Delta\text{Primal} \leq \alpha(\Delta\text{Dual})$ so we better design primal alg. s.t. $\Delta\text{Primal} \leq \begin{cases} \alpha & \text{if } i \leq B \\ \emptyset & \text{if } i > B. \end{cases}$

When $i > B$ we need $\Delta \text{Primal} = 0$

hence $x_i = x_{i-1} = 0$.

Recall $x_{i-1} + z \geq 1$ at time $i-1$

$\Longrightarrow$ z ~~musts~~ must equal 1 when we reach iteration $i-1 = B$ if not earlier.

What about making $\Delta \text{Primal} \leq \alpha$ when $i \leq B$?

The algorithm we're designing sets some value for x_i if the ski season lasts for $\geq i$ days.

Call this value $f\left(\frac{i}{B}\right)$ where $f : [0,1] \to [0,1]$.

$$\alpha \geq \Delta \text{Primal} = x_i + B(x_{i-1} - x_i)$$

$$= f\left(\tfrac{i}{B}\right) - B\left(f\left(\tfrac{i}{B}\right) - f\left(\tfrac{i-1}{B}\right)\right)$$

$$= f\left(\tfrac{i}{B}\right) - \frac{f(i/B) - f\left(\frac{i-1}{B}\right)}{1/B}$$

What about taking f to be a solution to

$$f(x) - f'(x) = \alpha.$$

With boundary condition $f(0) = 1$, $f(1) = 0$

the solution is

$$\alpha = \frac{e}{e-1} \qquad f(x) = 1 - (\alpha - 1)(e^x - 1).$$

Online matching

PRIMAL

$$\max \sum_{\{u,v\} \in E} x_{uv}$$

s.t. $\forall u \quad \sum_{v \in N(u)} x_{uv} \leqslant 1$

$\forall v \quad \sum_{u \in N(v)} x_{uv} \leqslant 1$

$x_{uv} \geqslant 0 \quad \forall u,v$

DUAL

$$\min \sum_{u \in L} y_u + \sum_{v \in R} y_v$$

s.t. $y_u + y_v \geqslant 1$

$\forall \text{edge } \{u,v\}$

$\vec{y} \geq 0$