11 Sep 2024 The Ski Rental Problem

Reminders: Prob Set 1 due Fri
(grace period Sun)
Hand in on Gradescope.
Each group submits one set of sol's.
Linda covers my OH today, 3:30-5, Gates 317.

SKI RENTAL

A skier will make T ski trips this season.
Each trip, they must either
- rent ← cost of 1 per trip
- own ← cost of B the first time you choose this option, ∅ after.

Goal: minimize total cost.

Offline version. T known, design optimal plan.

If T<B: buy once at start
T>B: rent each time
T=B: doesn't matter.

Online version: T starts at ∅ and is incremented by 1 an unknown # of times.

Online algorithm: Rent the first B times, then buy.

2-competitive:

$T \le B$ ALG pays T
OPT $= T$

$T > B$ ALG pays $2B$
OPT $= B$.

Deterministic can't be c-competitive for $c < 2$:

if the algorithm transitions to owning after renting k times, the worst case input has $T = k+1$, and the lower bound $c \ge 2$ follows by case analysis: $k < B$ vs. $k \ge B$.

For randomized algorithm, we will say it is c-competitive if...

$$\sup_T \frac{E[\text{algorithm's cost on input } T]}{OPT(T)} \le c.$$

LP relaxation of Ski rental

$$\min \quad \sum_{i=1}^{T} x_i + Bz$$

$$\text{s.t.} \quad x_i + z \geq 1 \quad \text{for } i = 1, \ldots, T$$

$$x_1, \ldots, x_T, z \geq 0$$

Dual

$$\max \quad \sum_{i=1}^{T} y_i$$

$$\text{s.t.} \quad y_i \leq 1 \quad \forall i$$

$$\sum_{i=1}^{T} y_i \leq B$$

$$y_i \geq 0 \quad \forall i$$

Online algorithm design strategy:

1. Set $\{x_i's, z\}$ and $y_i's$ in tandem.

x_t, y_t will be set permanently at time t.

z will only increase over time.

2. For all $t \leq T$ make sure

$(x_1, \ldots x_t, z)$ feasible for LP(t)

$(y_1, \ldots, y_t)$ feasible for DUAL(t)

3) Make sure PRIMAL LP objective never exceeds DUAL objective by more than a factor of

$$\alpha = \frac{e}{e-1}.$$